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Optimal Linear Estimators for Discrete-time Systems with One-step Random Delays and Multiple Packet Dropouts

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Abstract A novel model is developed to describe possible one-step random delay and multiple packet dropouts by two Bernoulli distributed random variables. Based on the developed model, the optimal linear estimators including filter, predictor and smoother for the state are presented in the linear minimum variance sense. The solution to the optimal filter is given in terms of a Riccati equation and a Lyapunov equation. The stability of the filter is analyzed. A sufficient condition for the existence of the steady-state filter is given.

Key words Estimator, random delay, packet dropout, linear minimum variance

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Recently, networked control systems and sensor networks have gained a lot of attention[1−5]. In networked systems, random delays, missing measurements and packet dropouts usually exist in data transmission through unreliable channels. So, it is significant to design the estimators and/or controllers over networks[5].

In networked systems, the phenomena of random delays, packet dropouts and missing measurements (or uncertain observations) can be described by stochastic parameters[6−19]. For systems with random delays, Ray et al.[6] present a suboptimal filter since the proposed filter is fixed to Kalman-like form. Yaz et al.[7] design a suboptimal filter by treating a colored noise as a white noise. Based on the covariance information method, the recursive linear estimator in the least-square criterion is solved[8]. For systems with missing measurements, [9−11] study the optimal linear estimation algorithms in least mean square sense. The corresponding robust filtering algorithms are also investigated for systems with random delays or missing measurements[12−13]. For systems with packet dropouts, a steady-state H_2 filter is presented by linear matrix inequality (LMI) in [14]. The similar filter is also designed for systems with packet dropouts of both sides from a sensor to a controller and from a controller to an actuator[15]. For a system with multiple packet dropouts similar to [14], the least mean square optimal linear estimators by the innovation analysis approach[16] and the full- and reduced-order linear estimators[17] by completion of square approach are designed, respectively. However, the random delays are not taken into account in [15−17].

So far, the aforementioned results are mainly focused on random delays or packet dropouts. In the recent study[18], a Kalman-like filter is designed for systems with one-step delays and multiple packet dropouts by completion of square. Further, the optimal linear estimators are presented for systems with bounded random delays and packet dropouts by the innovation analysis approach. However, the results in [18−19] can bring network congestion since each packet at sensor side is repeatedly transmitted several times to avoid data loss as far as possible.

In this paper, a new model is presented to describe possible one-step delay and multiple packet dropouts. Different from [18−19], here a packet at sensor side is only transmitted one time to avoid network congestion. Two Bernoulli random variables with known probabilities are employed to describe the random delays and packet dropouts. By introducing new variables, the system with random delays and packet dropouts is converted to a system with random parameters. The optimal linear estimators are obtained in terms of a Riccati equation and a Lyapunov equation parameterized by the distributions of two Bernoulli random variables. A sufficient condition of the existence for the steady state is given. For the cases of no delays and packet dropouts, only one-step delays or packet dropouts, the corresponding results can be obtained as the special cases of this paper.

1 Problem formulation

Consider the discrete-time linear stochastic system:

$$\boldsymbol{x}(t+1) = \Phi\boldsymbol{x}(t) + \Gamma\boldsymbol{w}(t) \tag{1}$$

$$\boldsymbol{z}(t) = H\boldsymbol{x}(t) + \boldsymbol{v}(t) \tag{2}$$

where $\boldsymbol{x}(t) \in \mathbf{R}^n$ is the state, $\boldsymbol{z}(t) \in \mathbf{R}^m$ is the sensor measurement, $\boldsymbol{w}(t) \in \mathbf{R}^r$ and $\boldsymbol{v}(t) \in \mathbf{R}^m$ are correlated white noises with zero-mean and variances $Q_{\boldsymbol{w}} \geq 0$, $Q_{\boldsymbol{v}} > 0$ and cross-covariance S, and Φ, Γ and H are constant matrices with appropriate dimensions. The initial state $\boldsymbol{x}(0)$ with mean $\boldsymbol{\mu}_0$ and variance P_0 is uncorrelated with $\boldsymbol{w}(t)$ and $\boldsymbol{v}(t)$.

We assume that there exist possible one-step random delay and multiple packet dropouts in the data transmission. To avoid network congestion, a packet at sensor side is only sent one time and the estimator only receives a packet at each time. The following model is adopted to describe the measurement received by the estimator:

$$\begin{aligned}\boldsymbol{y}(t) = \xi(t)\boldsymbol{z}(t) + (1-\xi(t))(1-\xi(t-1))\gamma(t)\boldsymbol{z}(t-1) + \\ (1-\xi(t))[1-(1-\xi(t-1))\gamma(t)]\boldsymbol{y}(t-1)\end{aligned} \tag{3}$$

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where $\xi(t)$ and $\gamma(t)$ are mutually uncorrelated Bernoulli distributed random variables with probabilities $P\{\xi(t)=1\}=\alpha$, $P\{\xi(t)=0\}=1-\alpha$, $P\{r(t)=1\}=\beta$, and $P\{r(t)=0\}=1-\beta$. where $0\leq\alpha\leq1$ and $0\leq\beta\leq1$, and they are also uncorrelated with other random variables.

Then, from (3) we have that the probability for a packet at time t received by the estimator on time is $P\{\xi(t)=1\}=\alpha$, the probability for one-step delay is $P\{\xi(t)=0,\ \xi(t+1)=0,\ \gamma(t+1)=1\}=(1-\alpha)^2\beta$, and the probability for packet dropout is $P\{\xi(t)=0,\ \xi(t+1)=1\}+P\{\xi(t)=0,\ \xi(t+1)=0,\ \gamma(t+1)=0\}=(1-\alpha)\alpha+(1-\alpha)^2(1-\beta)$ (also see the following Table 1). Moreover, we easily verify $\alpha+(1-\alpha)^2\beta+(1-\alpha)\alpha+(1-\alpha)^2(1-\beta)=1$.

Model (3) describes possible one-step random delay and multiple packet dropouts in the data transmission. The model shows that the latest measurement received by the estimator will be used for the estimation when the current packet is delayed or lost. It is clear that if $\xi(t)=0$ and $\xi(t-1)=1$ (or $\gamma(t)=0$), then $\boldsymbol{y}(t)=\boldsymbol{y}(t-1)$, which means that the received measurement at $t-1$ is used for the estimation. If $\xi(t)=0$, $\xi(t-1)=0$ and $\gamma(t)=1$, then $\boldsymbol{y}(t)=\boldsymbol{z}(t-1)$, which means the one-step delay. If $\xi(t)=1$, it means no delays and packet dropouts. Table 1 shows the data transmission case.

Table 1 Data transmission in network

t	1	2	3	4	5	6	7	8	9
$\xi(t)$	1	0	1	0	0	1	0	0	1
$\gamma(t)$					1		0	0	
$\boldsymbol{y}(t)$	$\boldsymbol{z}(1)$	$\boldsymbol{y}(1)$	$\boldsymbol{z}(3)$	$\boldsymbol{y}(3)$	$\boldsymbol{z}(4)$	$\boldsymbol{z}(6)$	$\boldsymbol{y}(6)$	$\boldsymbol{y}(7)$	$\boldsymbol{z}(9)$

From Table 1, we can see that $\boldsymbol{z}(1)$, $\boldsymbol{z}(3)$, $\boldsymbol{z}(6)$, and $\boldsymbol{z}(9)$ are received on time, $\boldsymbol{z}(4)$ is delayed one step, and $\boldsymbol{z}(2)$, $\boldsymbol{z}(5)$, $\boldsymbol{z}(7)$, and $\boldsymbol{z}(8)$ are lost.

Our aim is to find the optimal linear estimators $\hat{\boldsymbol{x}}(t|t+N)$ of state $\boldsymbol{x}(t)$ in the linear minimum variance sense where it is a filter if $N=0$, a predictor if $N<0$ and a smoother if $N>0$, based on the received measurements $(\boldsymbol{y}(t),\boldsymbol{y}(t-1),\cdots,\boldsymbol{y}(0))$. Note that here we will design the estimators that only depend on the distributions α and β but not the values of $\xi(t)$ and $\gamma(t)$. In this paper, the expectation E operates on $\xi(t)$, $\gamma(t)$, $\boldsymbol{w}(t)$, and $\boldsymbol{v}(t)$. 0 is a zero matrix with suitable dimensions.

2 Optimal linear estimators in the finite horizon

Let $\boldsymbol{Z}(t)=(1-\xi(t))\boldsymbol{z}(t)$ and $\boldsymbol{Y}(t)=\xi(t)\boldsymbol{y}(t)$, by using the result of $\xi(t)(1-\xi(t))=0$ and noting that the distributions of $\xi^2(t)$ and $\xi(t)$ are same, from (2) and (3) we have

$$\boldsymbol{Z}(t)=(1-\xi(t))H\boldsymbol{x}(t)+(1-\xi(t))\boldsymbol{v}(t)\tag{4}$$
$$\boldsymbol{Y}(t)=\xi(t)H\boldsymbol{x}(t)+\xi(t)\boldsymbol{v}(t)\tag{5}$$

Then systems (1) ~ (5) are equivalent to the following augmented system:

$$\boldsymbol{X}(t+1)=\tilde{\Phi}(t)\boldsymbol{X}(t)+\tilde{\Gamma}(t)\boldsymbol{W}(t)\tag{6}$$
$$\boldsymbol{y}(t)=\tilde{H}(t)\boldsymbol{X}(t)+\xi(t)\boldsymbol{v}(t)\tag{7}$$

where

$$\boldsymbol{X}(t+1)=\begin{bmatrix}\boldsymbol{x}^{\rm T}(t+1) & \boldsymbol{Z}^{\rm T}(t) & \boldsymbol{Y}^{\rm T}(t) & \boldsymbol{y}^{\rm T}(t)\end{bmatrix}^{\rm T}$$
$$\boldsymbol{W}(t)=\begin{bmatrix}\boldsymbol{w}^{\rm T}(t) & \boldsymbol{v}^{\rm T}(t)\end{bmatrix}^{\rm T}$$

and

$$\tilde{\Phi}(t)=\begin{bmatrix}\Phi & 0 & 0 & 0\\ \tilde{\xi}(t)H & 0 & 0 & 0\\ \xi(t)H & 0 & 0 & 0\\ \xi(t)H & \tilde{\xi}(t)\gamma(t)I_m & \tilde{\xi}(t)\gamma(t)I_m & \tilde{\xi}(t)(1-\gamma(t))I_m\end{bmatrix}$$
$$\tilde{\Gamma}(t)=\begin{bmatrix}\Gamma & 0\\ 0 & \tilde{\xi}(t)I_m\\ 0 & \xi(t)I_m\\ 0 & \xi(t)I_m\end{bmatrix}$$
$$\tilde{H}(t)=\begin{bmatrix}\xi(t)H & \tilde{\xi}(t)\gamma(t)I_m & \tilde{\xi}(t)\gamma(t)I_m & \tilde{\xi}(t)(1-\gamma(t))I_m\end{bmatrix}\tag{8}$$

where $\tilde{\xi}(t)=1-\xi(t)$.

For systems (6) and (7), the following statistical information holds.

$$Q={\rm E}[\tilde{\Gamma}(t)\boldsymbol{W}(t)\boldsymbol{W}^{\rm T}(t)\tilde{\Gamma}^{\rm T}(t)]=\begin{bmatrix}\Gamma Q_{\boldsymbol{w}}\Gamma^{\rm T} & (1-\alpha)\Gamma S & \alpha\Gamma S & \alpha\Gamma S\\ (1-\alpha)S^{\rm T}\Gamma^{\rm T} & (1-\alpha)Q_{\boldsymbol{v}} & 0 & 0\\ \alpha S^{\rm T}\Gamma^{\rm T} & 0 & \alpha Q_{\boldsymbol{v}} & \alpha Q_{\boldsymbol{v}}\\ \alpha S^{\rm T}\Gamma^{\rm T} & 0 & \alpha Q_{\boldsymbol{v}} & \alpha Q_{\boldsymbol{v}}\end{bmatrix}$$
$$\bar{S}={\rm E}[\tilde{\Gamma}(t)\boldsymbol{W}(t)\boldsymbol{v}^{\rm T}(t)\xi(t)]=\alpha\begin{bmatrix}S^{\rm T}\Gamma^{\rm T} & 0 & Q_{\boldsymbol{v}} & Q_{\boldsymbol{v}}\end{bmatrix}^{\rm T}\tag{9}$$

We easily obtain the following expectations:

$$\bar{\Phi}={\rm E}[\tilde{\Phi}(t)],\quad \bar{\Gamma}={\rm E}[\tilde{\Gamma}(t)],\quad \bar{H}={\rm E}[\tilde{H}(t)]\tag{10}$$

which can be given by (8) where $\xi(t)$ and $\gamma(t)$ are replaced by their expectations α and β.

Remark 1. From (3), we can see that a complicated derivation will be involved due to the correlated terms when the projection property is applied based on model (3) directly. To avoid the complicated derivation, by introducing new variables $\boldsymbol{Z}(t)$ and $\boldsymbol{Y}(t)$, the original systems (1) ~ (3) with one-step delay and packet dropouts is converted to the augmented systems (6) and (7) with stochastic parameters. We can easily apply projection property to derive our main results (see Theorems 1 ~ 3) in this paper. On the other hand, the augmented systems (6) and (7) have the dimension number of $n+3m$. In general, $n\geq m$, our augmented method can be adopted when there is large difference between n and m.

The following two preliminary lemmas are required to obtain the main results of this paper.

Lemma 1. For system (6), the state covariance matrix $q(t)={\rm E}[\boldsymbol{X}(t)\boldsymbol{X}^{\rm T}(t)]$ is computed by

$$q(t+1)=\Phi_0q(t)\Phi_0^{\rm T}+\alpha\Phi_0q(t)\Phi_1^{\rm T}+(1-\alpha)\beta\Phi_0q(t)\Phi_2^{\rm T}+\alpha\Phi_1q(t)\Phi_0^{\rm T}+\alpha\Phi_1q(t)\Phi_1^{\rm T}+(1-\alpha)\beta\Phi_2q(t)\Phi_0^{\rm T}+(1-\alpha)\beta\Phi_2q(t)\Phi_2^{\rm T}+Q\tag{11}$$

with the initial value

$$q(0)=\begin{bmatrix}P_0+\boldsymbol{\mu}_0\boldsymbol{\mu}_0^{\rm T} & 0\\ 0 & 0\end{bmatrix}$$

where

$$\Phi_0=\begin{bmatrix}\Phi & 0 & 0 & 0\\ H & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & I_m\end{bmatrix},\quad \Phi_1=\begin{bmatrix}0 & 0 & 0 & 0\\ -H & 0 & 0 & 0\\ H & 0 & 0 & 0\\ H & 0 & 0 & -I_m\end{bmatrix}$$