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Gradient geodesic and Newton geodesic HMP algorithms for the optimization of hybrid systems ☆,☆☆

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This paper provides algorithms for the optimization of autonomous hybrid systems based on the geometrical properties of switching manifolds. By employing the notion of geodesic curves on switching manifolds, the Hybrid Maximum Principle (HMP) algorithm introduced in Shaikh and Caines (2007) is extended to the so-called gradient geodesic and Newton geodesic algorithms. The convergence analysis for the algorithms is based upon the Lasalle Invariance Principle and simulation results illustrate their efficacy.

1. Introduction

The problem of hybrid systems optimal control (HSOC) has been studied and analyzed in many papers (see for instance Bengea & DeCarlo, 2005; Branicky, Borkar, & Mitter, 1998; Clarke & Vinter, 1989a, 1989b; Dmitruk & Kaganovich, 2008; Garavello & Piccoli, 2005; Grammel, 1999; Reidinger, Iung, & Krutz, 1999; Shaikh & Caines, 2007; Sussmann, 1999; Xu & Antsaklis, 2004). In particular, (Azhmyakov, Attia, & Raisch, 2008; Garavello & Piccoli, 2005; Shaikh & Caines, 2007; Sussmann, 1999) present an extension of the Maximum Principle to hybrid systems and (Shaikh & Caines, 2007) presents an iterative algorithm which is based upon the Hybrid Maximum Principle (HMP) necessary conditions for optimality. The HMP algorithm, presented in Shaikh and Caines (2007) for both autonomous and controlled switchings, is based upon a gradient search method for finding optimal switching states and times on switching manifolds. This paper is concerned with the optimal control problem for autonomous hybrid systems in terms of the switching states on switching manifolds and their associated switching times.

Following the approach introduced in Taringoo and Caines (2010, 2009), we apply Riemannian geometric methods (see Alvarez, Bolte, & Munier, 2004; Gabay, 1982; Mahony & Manton, 2008; Smith, 1994; Yang, 2007) based upon the generalization of steepest descent methods in Euclidean spaces in Gabay (1982) and Luenberger (1972). In contrast to the constrained optimization methods of Luenberger (1972), Mayne and Polak (1976) and Polak and Mayne (1976), a key feature of the formulation in Taringoo and Caines (2010, 2009) is that the iterative steps of the optimization algorithms occur within the (manifold) constraint subspaces.

In Section 2 optimal hybrid systems are introduced and Section 3 deals with the analysis of the Hybrid Maximum Principle algorithm. In Sections 4 and 5, specifically the HMP algorithm is generalized to the so-called gradient geodesic algorithm by employing the notion of geodesic curves on switching manifolds together with the methods introduced in Gabay (1982), Luenberger (1972), Yang (2007). The convergence analysis for the proposed algorithm is based on the Lasalle Invariance Principle (La Salle, 1976). In Section 6 in order to further improve the convergence rate, the so-called Newton geodesic version of the gradient geodesic algorithm is formulated in the local coordinate system of the switching state. Again the Lasalle Invariance Principle provides a proof of convergence for the Newton geodesic method. Simulation results show a significant improvement in terms of convergence rate and stability compared with the HMP algorithm.

2. Hybrid systems

The standard hybrid systems framework (Branicky et al., 1998; Shaikh & Caines, 2007; Sussmann, 1999) is as follows:

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Definition 1. A *hybrid system* is a six tuple

$$\mathbf{H} := \{H = Q \times \mathbf{R}^{\mathbf{n}+1}, \Gamma A, I = \Sigma \times U, F, \mathbf{M}\} \quad (2.1)$$

satisfying:

A0: $Q = \{1,2,3,\ldots,|Q|\}$ is the finite set of *discrete states*.

H is the *hybrid state space* of $\mathbf{H}$.

Γ: $H \times \Sigma \to Q$ is the time independent (partially defined) *discrete transition map*.

A: $Q \to 2^Q$ is the *set valued function* for which for a state $q \in Q$ all and only discrete controlled transitions in to the q dependent subset $A(q) \subset Q$ are allowed under Γ.

$\Sigma = \Sigma_u \dot\cup \Sigma_c \dot\cup \{id\}$ is a finite set of distinct autonomous (i.e. uncontrolled) and controlled *discrete event transition labels* extended with the identity element $\{id\}$ such that for $i \in Q, \sigma_{i,j} \in \Sigma$ only if $j \in A(i)$.

$U \subset R^u$ is the set of *admissible input control values*, where U is an open bounded set in R^u. The set of *admissible input control functions* is $\mathcal{U} := \mathcal{U}(U, L_\infty[0, T_*))$, the set of all bounded measurable functions on some interval $[0, T_*)$, $T_* \leqslant \infty$, taking values in U.

$I := \Sigma \times U$ is the set of *system input values*.

F is the *indexed collection of vector fields* $\{f_j\}_{j \in Q}$ such that f_j: $R^{n+1} \times U \to R^{n+1}$ is a uniform Lipschitz vector field assigned to each location.

We assume there exists $K_f < \infty$ such that $max_{j \in Q} sup_{u \in U} \|f_j(0,u)\| \leqslant K_f$, $u \in U$, $j \in Q$.

A *switching manifold* or guard $m_{p,q}$ is the union (over k) of a set of switching manifold components $m^k_{p,q} = \bigcup_{k_i; 1 \leqslant i \leqslant n(k)} \tilde m^{k_i}_{p,q}$, $\tilde m^{k_i}_{p,q} \in \mathcal{M}$, where $\mathcal{M} := \left\{\tilde m^k_\gamma : \gamma \in Q \times Q, k \in \mathbf{Z}_+\right\}$ is a collection of time independent manifold subcomponents such that for any ordered pair $\gamma = (p,q), \tilde m^k_\gamma$ is a smooth, i.e. C^∞, codimension 1 submanifold of R^{n+1}, possibly with boundary $\partial \tilde m^k_\gamma$. By abuse of notation, in the case of embedded submanifolds (in R^{n+1}), we describe the manifold subcomponents locally by $\tilde m^k_\gamma = \left\{x : \tilde m^k_\gamma(x) = 0\right\}$.

In this paper it is assumed that:

(i) $x \in \tilde m^{k_i}_\gamma$ is such that $x \in \tilde m^{k_i}_\gamma \bigcap \tilde m^{k_j}_\gamma, k_i \neq k_j$, if and only if $x \in \partial \tilde m^{k_i}_\gamma \bigcap \partial \tilde m^{k_j}_\gamma$.

(ii) If $\partial \tilde m^{k_i}_\gamma \bigcap \partial \tilde m^{k_j}_\gamma \neq \emptyset$ then $\partial \tilde m^{k_i}_\gamma \bigcap \partial \tilde m^{k_j}_\gamma$ is a pieces wise smooth codimention 2 submanifold of R^{n+1} (possibly with boundary).

(iii) For all $\gamma \in Q \times Q$, the family of switching manifolds subcomponents intersections are assumed to be locally finite. □

It should be noted that if $\tilde m^k_\gamma(.) \in C^\alpha(R^{n+1})$, the Implicit Function Theorem implies that the zero level set of $\tilde m^k_\gamma(x)$, i.e. $x \in R^{n+1}$ s.t. $x \in \tilde m^{k^{-1}}_\gamma(0)$, is locally given by $(y, \hat m(y))$, $y \in R^n$, where $m^k_\gamma(x)$ and $\hat m$ both have the same degree of regularity α, (Lee, 2002). In this paper the analysis will be assumed to be restricted to a single manifold subcomponent which is denoted by M.

A1: The *initial state* $h_0 := (x(t_0), q_0) \in \mathbf{H}$ is such that $m_{q_0 q_j}(x_0) \neq 0$ for all $q_j \in Q$. It is assumed that for all p, q, whenever a trajectory governed by the controlled vector field f_p meets any given guard manifold $m_{p,q}$ transversally, there is an autonomous switching to the controlled vector field f_q, also transversal to $m_{p,q}$, otherwise a continuation of the system trajectory is not defined.

Definition 2. A *hybrid system input* is a triple $I := (\tau, \sigma, u)$ defined on a half open interval $[t_0, T), T \leqslant \infty$, where $u \in \mathcal{U}$ and (τ, σ) is a *hybrid switching sequence* $(\tau, \sigma) = ((t_0, \sigma_0), (t_1, \sigma_1), (t_2, \sigma_2), \ldots)$, $t_0 < t_1, \ldots$, of pairs of switching times and discrete input events, $\sigma_0 = id$, $\sigma_i \in \Sigma$, $i \geqslant 1$, where σ is called a *location sequence*. The corresponding *hybrid state trajectory* is a triple (τ, q, x) consisting of τ, an associated sequence of discrete states $q = (q_0, q_1, q_2, \ldots)$, and a sequence $x(\cdot) = (x_{q_0}(\cdot), x_{q_1}(\cdot), x_{q_2}(\cdot), \ldots)$ of absolutely continuous functions $x_{q_j} : [t_j, t_{j+1}) \to R^{n+1}$. □

Let $\{l_j\}_{j \in Q}$, $l_j \in C^k(R^{n+1} \times U; R_+)$, $k \geqslant 1$, be a family of loss functions and $h \in C^k(R^{n+1}; R_+)$, $k \geqslant 1$, a terminal cost satisfying the following hypothesis:

A2: There exist $K_l < \infty$ and $1 \leqslant \gamma < \infty$ such that $|l_j(x,u)| \leqslant K_l(1 + \|x\|^\gamma)$, $x \in R^{n+1}$, $u \in U$, $j \in Q$, and similarly for $h(.)$.

Consider the initial time t_0, final time $t_f < \infty$, initial hybrid state $h_0 = (q_0, x_0)$, and $\bar L < \infty$. Let $S_L = ((t_0, \sigma_0), (t_1, \sigma_1), \ldots, (t_L, \sigma_L))$ be a hybrid switching sequence and let $I_L := (S_L, u), u \in \mathcal{U}$, be a hybrid input trajectory subject to ***A0, A1***, where $L \leqslant \bar L < \infty$, is the number of switchings. Subject to ***A2***, define the *hybrid cost function* as

$$J(t_0, t_f, h_0; I_L, \mathcal{U}) := \sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s), u(s)) + h(x_{q_L}(t_f)). \quad (2.2)$$

The continuous dynamics of the hybrid system are specified as follows:

$$\begin{aligned}
&\dot x_{q_i}(t) = f_{q_i}(x_{q_i}(t), u(t)), \quad a.e. t \in [t_i, t_{i+1}),\\
&u(t) \in U \subset R^u, u(.) \in L_\infty(U), \quad h_0 = (q_0, x_0), \quad i = 0, 1, \ldots, L,\\
&x_{q_{i+1}}(t_{i+1}) = lim_{t \to t_{i+1}} x_{q_i}(t), \quad t_{L+1} = t_f < \infty. \quad (2.3)
\end{aligned}$$

It should be noted that, in general, different controls result in different sequence of dynamics and numbers (L) of switchings (Shaikh & Caines, 2007). In this paper it is assumed that the minimization of (2.2) is performed in a class of control functions, $\mathcal{U}$, which generate a given aprior sequence of discrete transition events σ_i, $i = 1, \ldots, L$. In addition it is assumed that all optimal switching states corresponding to the minimization of the cost defined in (2.2) lie in the interior of switching manifolds subcomponents.

The following theorem gives the Hybrid Maximum Principle in the extended class of the cases treated in Shaikh and Caines (2007), specifically the autonomous switchings case is extended to the time varying guards case. It is shown that the discontinuity of the Hamiltonian functions and adjoint variables at the optimal switching state and switching time give important information about the geometry of the switching manifold M at switching states.

Theorem 1. *Consider a hybrid system satisfying the assumptions* **A0–A2** *above and define*

$$H_q(x, \sigma, u, \lambda) = \lambda^T f_{\sigma(q)}(x, u) + l_{\sigma(q)}(x, u), \lambda \in R^{n+1}, \ u \in U, q \in Q.$$

Assume that I_L contains only autonomous switchings and let

$$J^o(t_0, t_f, h_0, \mathcal{U}) = inf_{I_L} J(t_0, t_f, h_0, I_L, \mathcal{U})$$

be the infimized cost function with infimizing control I^o_L and trajectory (x^o, q^o) which are both assumed to exist. Let I^o_L have L autonomous switchings and let t_1, t_2, …, t_L, denote the switching times along the optimal trajectory.

Finally, assume that almost everywhere along an optimal trajectory the continuous state x satisfies the controllability condition given in Shaikh and Caines (2007). Then:

(i) *There exists a piecewise absolutely continuous adjoint process satisfying*

$$\dot\lambda^o_j = -\frac{\partial H_j}{\partial x}(x^o, \sigma^o, \lambda, u^o), \quad u^o_t \in U \ a.e., \ t \in (t_j, t_{j+1}).$$

(ii) *At the switching times the adjoint process and Hamiltonain function satisfy*

$$\lambda_j\left(t_j^-\right) = \lambda_{j+1}\left(t_j^+\right) + p_j \nabla_x m_{j,j+1}(x(t_j), t_j), \quad 1 \leqslant j \leqslant L, \quad (2.4)$$

$$H_j\left(t_j^-\right) = H_{j+1}\left(t_j^+\right) - p_j \nabla_t m_{j,j+1}(x(t_j), t_j), \quad 1 \leqslant j \leqslant L. \quad (2.5)$$