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On the role of optimality functions in numerical optimal control ☆

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ABSTRACT

We present a survey of optimality conditions in optimality function form and discuss their role in establishing that discretized optimal control problems are consistent approximations to the original optimal control problems.

1. Introduction

The numerical solution of optimal control problems requires the discretization of the dynamics. This can be done using the Euler method, see, for example Polak (1997), the Runge–Kuta method, as in Schwartz and Polak (1996), both of which lead to dynamics in difference equation form, or using a collocation technique, such as the one using pseudo-spectral factorization techniques, see, e.g. Betts (2009), Gong, Kang, and Ross (2006), Kang, Gong, and Ross (2007), which approximate the control and states by polynomial expansions. In either case, the resulting approximating problems are nonlinear programming problems. An important question that must be answered when a discretization scheme is proposed is whether, as the discretization is refined, the optimal solutions of the discretized problems converge to optimal solutions of the original optimal control problem, and whether the stationary points of discretized problems converge to stationary points of the original optimal control problem.

Then it is true that the optimal solutions of the discretized problems converge to optimal solutions of the original optimal control problem, and the stationary points of discretized problems converge to stationary points of the original optimal control problem, as the discretizations are refined, we will say that the discretized problems are *consistent approximations* to the original problem.

Sufficient conditions for the convergence of the optimal solutions of the discretized problems to solutions of the original problem can be established within the framework of *epiconvergence*, see, e.g., Polak (1997), Rockafellar and Wets (1997). Depending on the specific discretization scheme chosen, this may be a relatively straightforward task, as in the case Euler discretizations, see Polak (1997), or technically quite challenging, as in the case of spectral factorizations (Kang et al., 2007).

Stationary points are points that satisfy a first-order optimality condition. Showing that stationary points of finite dimensional discretized optimal control problems converge to stationary points of the original, infinite dimensional optimal control problem, requires compatibility of the characterization of their stationary points. The classical optimality conditions for nonlinear programming problems are the Karush–Kuhn–Tucker and the F. John conditions (see, e.g. Polak, 1997), while in the case of an infinite number of constraints (as in the case of trajectory constraints), i.e., semi-infinite optimization problems, in a form that involves the subgradients of max functions, see, e.g., Polak (1997). Stationary points of optimal control problems are usually characterized in terms of the Pontryagin Maximum principle in its various forms, see, e.g., Pontryagin, Boltianski, Gramkrelidze, and Mischenko (1962), Vinter (2000). Since there is no such thing as Pontryagin variations in finite space, it quickly becomes clear that the above mentioned optimality conditions are incompatible. Hence it is necessary to use optimality conditions which, formally, have the same structure for both finite and infinite dimensional problems. As we will see, *optimality functions* turn out to be the perfect tool for this purpose.

After briefly considering the concept of consistent approximations, we will proceed with our primary task: that of surveying optimality functions for numerical optimal control.

2. Problem in abstract form

Let S be a normed space, let $f^0 : S \to \mathbb{R}$ be a cost function, and let $C \subset S$ be a constraint set, where $S = C$ is possible. Consider the infinite dimensional problem

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$$\mathbf{P} \qquad \min\{f^0(z)|z \in C \subset S\}. \tag{1a}$$

Next, for $i = 1, 2, \ldots$, let $S_i \subset S$ be a sequence of nested finite dimensional subspaces of S, with the dimension of S_i growing to infinity as $i \to \infty$, and $\cup_{i\in\mathbb{N}} S_i$ dense in S.

Next, let $f_i^0 : S_i \to \mathbb{R} \to \infty$ be a sequence of cost functions, and let $C_i \subset S_i$, $i = 1, 2, 3, \ldots$ be a sequence of finite dimensional constraint sets. We can now define the sequence of finite dimensional approximating problems $\mathbf{P}_i$, $i = 1, 2, 3, \ldots$, as follows,

$$\mathbf{P}_i \qquad \min\{f_i^0(z)|z \in C_i\}. \tag{1b}$$

Definition 1. We say that the sequence of problems $\{\mathbf{P}_i\}_{i=0}^{\infty}$ is a sequence of *consistent approximations* to the problem $\mathbf{P}$ if the global minimizers of the $\mathbf{P}_i$ converge to global minimizers o $\mathbf{P}$ and the stationary points (which can be local minimizers) of the $\mathbf{P}_i$ converge to stationary points (which can be local minimizers) of $\mathbf{P}$. □

A sufficient condition for global minimizers of the $\mathbf{P}_i$ to converge to global minimizers of $\mathbf{P}$ is given by the concept of *epiconvergence*.

Definition 2.

(a) We define the *epigraphs* of f^0, restricted to C, and of f_i^0, restricted to C_i, by

$$Epi(f^0|C) \triangleq \{(z^0, z) \in \mathbb{R} \times C | z^0 \geqslant f^0(z)\} \tag{2a}$$

$$Epi(f_i^0|C_i) \triangleq \{(z^0, z) \in \mathbb{R} \times C_i | z^0 \geqslant f_i^0(z)\} \tag{2b}$$

(b) We say that the problems $\mathbf{P}_i$ *epiconverge* to the problem $\mathbf{P}$, if $Epi(f_i^0|C_i) \to Epi(f^0|C)$, as $i \to \infty$, i.e.,
  (i) For any sequence $\{\bar{z}_i\}_{i=0}^{\infty}$ such that $\bar{z}_i \in Epi(f_i^0|\mathbf{C}_i)$, if $\bar{z}_i \to \bar{z}^*$, as $i \to \infty$, then $\bar{z}^* \in Epi(f^0|\mathbf{C})$, and
  (ii) for any $\bar{z} \in Epi(f^0|\mathbf{C})$ there exists a sequence $\{\bar{z}_i\}_{i=0}^{\infty}$, with $\bar{z}_i \in Epi(f_i^0|\mathbf{C}_i)$, such that $\bar{z}_i \to \bar{z}$. □

The above definition is a bit awkward to use in practice and hence one often prefers to rely on the following alternative characterization, see Polak (1997), Rockafellar and Wets (1997):

Theorem 3. *The epigraphs* $Epi(f_i^0|C_i)$, $i = 1, 2, 3, \ldots$ *converge to the epigraph* $Epi(f^0|C)$, *if and only if*

(a) *for every* $z \in \mathbf{C}$, *there exists a sequence* $\{z_i\}_{i\in\mathbb{N}}$, *with* $z_i \in C_i$, *such that* $z_i \to z$, *as* $i \to \infty$; *and* $\limsup f_i^0(z_i) \leqslant f^0(z)$;
(b) *for every infinite sequence* $\{z_i\}_{i\in K}$, *with* $K \subset \mathbb{N}$, *such that* $z_i \in C_i$, *for all* $i \in K$, *and* $z_i \to^K z$, *as* $i \to \infty$, $z \in C$; *and* $\liminf_K f_i^0(z_i) \geqslant f^0(z)$. □

Fig. 1 illustrates epiconvergence.

Theorem 4. *If the problems* $\mathbf{P}_i$ *epiconverge to* $\mathbf{P}$ *and* $\hat{z}_i$ *is a global minimizer of* $\mathbf{P}_i$, $i = 1, 2, 3, \ldots$, *such that* $\hat{z}_i \to \hat{z}$, *then* $\hat{z}$ *is a global minimizer of* $\mathbf{P}$. *Furthermore, the optimal values* $f_i^0(z_i)$, *of the* $\mathbf{P}_i$, *converge to the optimal value* $f^0(\hat{z})$ *of* $\mathbf{P}$. □

Thus, epiconvergence provides a convenient sufficient condition for ensuring that global minimizers and global minimum values of the approximating problems converge to global minimizers and the global minimum value of the original problem.

Next we turn to local minimizers, or, more exactly, stationary points. Proofs of results, quoted below, can be found in Polak et al. (1993), and Polak (1997).

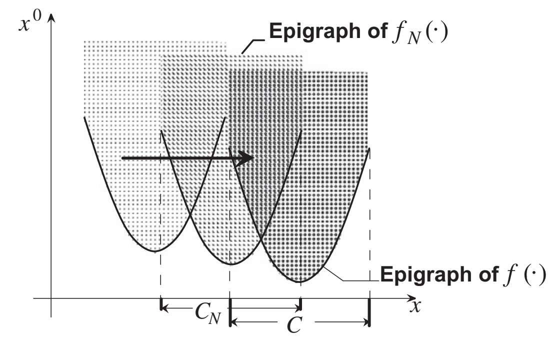

Fig. 1. Convergence of approximating epigraphs.

Definition 5. Let $\mathbb{R}_- \triangleq \{y \in \mathbb{R} | y \leqslant 0\}$.

(i) If $\theta : \mathbf{S} \to \mathbb{R}_-$, $\theta_i : \mathbf{S}_i \to \mathbb{R}_-$, $i = 1, 2, 3, \ldots$ are continuous functions such that for any local minimizer $\hat{z}$ of $\mathbf{P}$, $\theta(\hat{\mathbf{z}}) = \mathbf{0}$, and for any local minimizer $\hat{z}_i$ of $\mathbf{P}_i$, $\theta_i(\hat{z}) = 0$, then we say that $\theta(\cdot)$, $\theta_i(\cdot)$ are *optimality functions*.
(ii) If $\hat{z} \in \mathbf{C}$ is such that $\theta(\hat{z}) = 0$, then we say that $\hat{z}$ is a *stationary point* of $\mathbf{P}$. If $\hat{z} \in \mathbf{C}_i$ is such that $\theta_i(\hat{z}) = 0$, then we say that $\hat{z}$ is a *stationary point* of $\mathbf{P}_i$. □

We can now make again use of the concept of epiconvergence, as follows:

Theorem 6. *Suppose that the functions* $\theta : \mathbf{S} \to \mathbb{R}_-$ *and* $\theta_i : \mathbf{S}_i \to \mathbb{R}_-$, $i = 1, 2, 3, \ldots$ *are optimality functions for* $\mathbf{P}$ *and* $\mathbf{P}_i$, *respectively.*

If $Epi(-\theta_i|S_i) \to Epi(-\theta|S)$ *and* $\hat{z}_i$ *is a stationary point of* $\mathbf{P}_i$, $i = 1, 2, 3, \ldots$, *such that* $\hat{z}_i \to \hat{z}$, *then* $\hat{z}$ *is a stationary point of* $\mathbf{P}$. □

In fact, one can show that if $\{z_i\}_{i\in\mathbb{N}}$ are such that $z_i \in C_i$, $z_i \to \hat{z}$, as $i \to \infty$, and $\theta_i(z_i) \to 0$, as $i \to \infty$, then $\theta(\hat{z}) = 0$, i.e., that progressively better approximations to stationary points of the $\mathbf{P}_i$ converge to a stationary point of $\mathbf{P}$.

3. Optimal control: Euler discretization

First we consider optimal control problems as optimization problems with cost functions and constraints depending only on the control variable u, since the state is determined by the control via the dynamics. We begin by defining a space for the controls, in which we are guided by two considerations. First, since controls are usually *pointwise* bounded, the controls must be in L_∞ and since we need to have gradients, which implies a Hilbert space, the space we adopt is a pre-Hilbert space which is a cross between L_∞ and L_2, as follows.

Let

$$L^m_{\infty,2}[0,1] \triangleq \left(L^m_\infty[0,1], \langle\cdot,\cdot\rangle_{L_2}, \|\cdot\|_{L_2}\right), \tag{3a}$$

We begin by defining cost and constraint functions.

Let $F^0 : \mathbb{R}^n \to \mathbb{R}$, $\Phi^j : \mathbb{R}^n \times [0,1] \to \mathbb{R}$, and $G : \mathbb{R}^n \to \mathbb{R}^r$ be C^1 functions. Let $f^0 : L^m_{\infty,2}[0,1] \to \mathbb{R}$ and $\phi^j : L^m_{\infty,2}[0,1] \times [0,1] \to \mathbb{R}$, $j \in \mathbf{q} \triangleq \{1, 2, \ldots, q\}$ be defined by

$$f^0(u) \triangleq F^0(\boldsymbol{x}^u(1)), \tag{3b}$$

$$\phi^j(u,t) \triangleq \Phi^j(\boldsymbol{x}^u(t), t), \tag{3c}$$

$$g(u) \triangleq G(x^{\mathbf{u}}(1)), \tag{3d}$$

where $x^u(t)$ is the solution of the differential equation

$$\dot{x}(t) = h(x(t), u(t)), \quad t \in [0,1], \quad x(0) = \xi, \tag{3e}$$