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Stability of piecewise linear systems revisited☆

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Abstract: Piecewise linear systems are important in representing and approximating many practical systems with complex dynamics. While stability analysis of switched linear systems are notoriously challenging, several powerful tools have been developed to cope with the challenges. This paper provides a brief survey on stability of piecewise linear systems. The approaches introduced here range from the Lyapunov method to switching-transition-based analysis, and the combined. Main features and advantages of each approach are discussed and compared. Numerical examples and a case study are also presented to illustrate the effectiveness of the approaches.

1. Introduction

Piecewise linear systems are switched linear systems with state-space-partition-based switching. Piecewise linear systems are very important in representing many practical systems including power electronics (Deane & Hamill, 1990; Guzelis & Goknar, 1991), biology regulatory networks (De Jong et al., 2004; Oktem, 2005), and social opinion dynamics (Krause, 2000; Lorenz, 2007), to list a few. Theoretically, piecewise linear systems are powerful in approximating highly nonlinear dynamic systems (Biswas, Grieder, Lofberg, & Morari, 2005; Nakamura & Hamada, 1990; Sontag, 1981), in representing interconnections of linear systems and finite automata (Sontag, 1996), and in characterizing control systems with fuzzy logics (Feng, 2004; Rovatti, 1998). From the model point of view, piecewise linear systems provide equivalent framework to the well-known linear complementary systems (Cottle, Pang, & Stone, 1992; Heemels, De Schutter, & Bemporad, 2001; van der Schaft & Schumacher, 1998) and mixed logical dynamical systems (Bemporad & Morari, 1999; Heemels, Schumacher, & Weiland, 2001).

While popular in modelling and powerful in control, piecewise linear systems are very hard to investigate. One major difficulty comes from the intrinsic discontinuities caused by autonomous switching among the subsystems, where the transitions over the switching surfaces are generally nonlinear and multi-valued. Besides, ill-posedness and sliding motions are issues that have to be addressed a priori in system analysis. Despite the difficulties and challenges, remarkable progress have been achieved thanks to the dedication of many researchers over the last decades. Important achievements include well-posedness (Imura, 2003; Xia, 2002; Camlibel et al., 2006), controllability and reachability (Arapostathis & Broucke, 2007; Asarin, Bournez, Dang, & Maler, 2000; Camlibel, Heemels, & Schumacher, 2008), stability and stabilization (Goncalves, Megretski, & Dahleh, 2003; Johansson & Rantzer, 1998), and computational complexity (Blondel & Tsitsiklis, 1999), among others.

In this work, we focus on stability analysis of piecewise linear systems, and provide a brief review of the major approaches in a systematic manner. More specifically, we present several approaches for analyzing stability of general piecewise linear systems, including the piecewise quadratic Lyapunov approach, the surface Lyapunov approach, and the transition graph approach. Numerical examples and a case study are provided to illustrate the effectiveness of the approaches.

2. System description

Notations. $\mathbf{R}$ is the set of real numbers, and $\mathbf{R}_+$ is the set of nonnegative real numbers. Let $|\cdot|$ be any norm on $\mathbf{R}^n$ and $\|\cdot\|$ the induced matrix norm. In particular, ℓ_p-norm will be denoted as $|\cdot|_p$ for $p \in [1, \infty]$. For any positive real number r, let

$$\mathbf{B}_r^n = \{x \in \mathbf{R}^n : |x| \le r\}$$

and

$$\mathbf{H}_r^n = \{x \in \mathbf{R}^n : |x| = r\}.$$

The superscript n will be dropped when no confusion occurs. For two sets S_1 and S_2, set $S_1 - S_2$ includes each element of S_1 that does not belong to S_2. Let $M = \{1, 2, \ldots, m\}$ be a finite index set.

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To set up the system framework, let $\Omega_1, \ldots, \Omega_m$ be a non-degenerate polytopic partition of the state space, that is, each region Ω_i is a (convex) polyhedron with nonempty interior, $\cup_{i=1}^{m}\Omega_i = \mathbf{R}^n$ and $\Omega_i \cap (\Omega_j)^o = \varnothing$ for $i \neq j$, where Ω^o denotes the interior of Ω with respect to $\mathbf{R}^n$. Then, a piecewise linear system is mathematically described by

$$x^+(t) = A_i x(t) + a_i, \qquad x(t) \in \Omega_i, \tag{1}$$

where $x \in \mathbf{R}^n$ is the state, $A_i \in \mathbf{R}^{n\times n}$, $a_i \in \mathbf{R}^n$, and x^+ denote the derivative operator in continuous time, and the shift forward operator in discrete time. The systems are known as piecewise affine systems in the literature due to the existence of the affine terms. When the affine terms vanish, the systems are said to be piecewise linear systems. Here we abuse the notation as a piecewise affine system can be converted into a piecewise linear system via expanding the system dimension by one. It is clear that the evolution of the continuous state relies only on the initial configuration, hence is denoted by $\phi(t;t_0, x_0)$. To avoid ill-posedness, we assume that the right-hand side is always continuous over the boundaries. Under this assumption, we further assume without loss of generality that partition cells Ω_i are closed polyhedra. To guarantee that the origin is an equilibrium, we assume that $a_i = 0$ when $0 \in \Omega_i$. For notational convenience, we divide the index set M into M_1 and M_2 such that $i \in M_1$ iff $0 \in \Omega_i$. Finally, for $i \in M_2$, denote $\bar{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix}$ in continuous time, and $\bar{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 1 \end{bmatrix}$ in discrete time.

It has been established that, piecewise linear systems are able to represent a mixed logical dynamical system (Bemporad & Morari, 1999)

$$\begin{aligned} &y^+(t) = Ay(t) + B_1\delta(t) + B_2 z(t) \\ &E_1 y(t) + E_2\delta(t) + E_3 z(t) + E_4 \succcurlyeq 0, \end{aligned} \tag{2}$$

where $y = \begin{bmatrix} y_c \\ y_d \end{bmatrix}$ with $y_c \in \mathbf{R}^{n_c}$ and $y_d \in \{0, 1\}^{n_d}$, $\delta \in \{0, 1\}^{r_d}$, $z(t) \in \mathbf{R}^{r_c}$ are auxiliary variables, and $A, B_1, B_2, E_1, \ldots, E_4$ are constant matrices and vectors with compatible dimensions. $\succcurlyeq$ stands for component-wise inequality. Similarly, piecewise linear systems are able to represent a linear complementarity system (Heemels et al., 2000; van der Schaft & Schumacher, 1998)

$$\begin{aligned} &y^+(t) = Ay(t) + Bw(t) \\ &v(t) = E_1 y(t) + E_2 w(t) + E_3 \\ &0 \preccurlyeq v(t) \perp w(t) \succcurlyeq 0, \end{aligned} \tag{3}$$

where $y(t) \in \mathbf{R}^n$, $w(t) \in \mathbf{R}^k$, $v(t) \in \mathbf{R}^k$, and $\perp$ denote the orthogonality of vectors.

The frameworks of piecewise linear systems, mixed logical dynamical systems, and linear complementarity systems were proposed with different backgrounds for representing specific subclasses of hybrid dynamical systems. Under mild conditions about well-posedness and feasibility, the subclasses are equivalent to each other, see Heemels et al. (2001) and Bemporad (2004) for details. As a result, the stability criteria stated in this paper also apply to mixed logical dynamical systems, and linear complementarity systems.

3. Piecewise quadratic Lyapunov approach

Note that, piecewise linear systems with origin equilibrium are locally (around the origin) piecewise linear with autonomous switching. As a result, any sufficient condition for stability under arbitrary switching is also sufficient for the autonomous stability. In particular, if the subsystems admit a common quadratic Lyapunov function, then the system is autonomously stable (Cheng, Guo, & Huang, 2003; Liberzon & Morse, 1999; Narendra & Balakrishnan, 1994). While a lot stability criteria were given by exploiting this idea, the criteria are doomed to be conservative as the partition information is ignored. A less conservative idea is to exploit the piecewise quadratic Lyapunov functions that are partition-dependent, which result in bilinear matrix inequalities or even linear matrix inequalities that are efficiently solvable numerically, as discussed below.

A piecewise quadratic Lyapunov function for the piecewise linear system is in the form

$$V(x) = x^T P_i x + 2q_i x + r_i = \bar{x}^T \bar{P} \bar{x}, \quad x \in \Omega_i, \tag{4}$$

where $\bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}$, $\bar{P}_i = \begin{bmatrix} P_i & q_i^T \\ q_i & r_i \end{bmatrix}$. When $i \in M_1$, we require that $q_i = 0$ and $r_i = 0$, which means that $V(x) = x^T P_i x$ for $x \in \Omega_i$. For consistency, we also re-define $\bar{x}$ to be $[x^T, 0]^T$ when $i \in M_1$. To apply the Lyapunov approach, we need to figure out the continuity of the function over the cell boundaries, and the definite positiveness of the function.

Fix a natural number k. Let $\bar{F}_i \in \mathbf{R}^{k\times(n+1)}$, $i = 1, \ldots, m$ be a sequence of matrices. The matrix sequence is said to be a continuity matrix sequence *w.r.t.* the partition sequence $\Omega_1, \ldots, \Omega_m$, if

$$\bar{F}_i \bar{x} = \bar{F}_j \bar{x}, \quad \forall x \in \Omega_i \cap \Omega_j, \ i \neq j.$$

Similarly, let $\bar{E}_i \in \mathbf{R}^{k\times(n+1)}$, $i = 1, \ldots, m$ be a sequence of matrices. The matrix sequence is said to be polyhedral cell bounding *w.r.t.* the partition sequence $\Omega_1, \ldots, \Omega_m$, if

$$\bar{\mathbf{E}}_i \bar{x} \succcurlyeq 0, \quad \forall x \in \Omega_i.$$

With the help of the above preparations, we are ready to state the main result on piecewise quadratic stability.

Theorem 3.1. (Johansson, 2003) *Let k be a natural number, $\bar{E}_i = [E_i, e_i]$ and $\bar{F}_i = [F_i, f_i]$, $i = 1, \ldots, m$, be a polyhedral cell bounding matrix sequence and a continuity matrix sequence, respectively, and T, U_i, W_i be symmetric matrices with $T \in \mathbf{R}^{k\times k}$, $U_i, W_i \in \mathbf{R}_+^{k\times k}$, $i = 1, \ldots, m$. Suppose that*

$$P_i = F_i^T T F_i, \quad i \in M_1$$

$$\bar{P}_i = \bar{F}_i^T T \bar{F}_i, \quad i \in M_2$$

satisfy

$$\left.\begin{aligned} &A_i^T P_i + P_i A_i + E_i^T U_i E_i < 0 \\ &P_i - E_i^T W_i E_i > 0 \end{aligned}\right. \ i \in M_1 \tag{5}$$

and

$$\left.\begin{aligned} &\bar{A}_i^T \bar{P}_i + \bar{P}_i \bar{A}_i + \bar{\mathbf{E}}_\mathbf{i}^\mathbf{T} \mathbf{U_i} \bar{\mathbf{E}}_\mathbf{i} < \mathbf{0} \\ &\bar{P}_i - \bar{\mathbf{E}}_\mathbf{i}^\mathbf{T} \mathbf{W_i} \bar{\mathbf{E}}_\mathbf{i} > \mathbf{0} \end{aligned}\right. \ i \in M_2. \tag{6}$$

Then, the piecewise linear system is globally exponentially stable.

Remark 3.1. Piecewise quadratic Lyapunov functions were used to analyze stability of continuous-time piecewise linear systems in Johansson and Rantzer (1998). Counterpart works for the discrete-time case could be found in (Ferrari-Trecate, Cuzzola, Mignone, & Morari, 2002) and Feng (2002).

Remark 3.2. The main advantages of the piecewise quadratic Lyapunov function approach include: (1) the criterion is much less conservative than the existence of a common quadratic Lyapunov function; (2) the searching of piecewise quadratic Lyapunov function is reduced to a set of linear matrix inequalities (LMIs),