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Lyapunov functionals and matrices☆

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ABSTRACT

In this contribution we present some basic results concerning the computation of quadratic functionals with prescribed time derivatives for linear time delay systems. Some lower and upper bounds for the functionals are given. The functionals are defined by special matrix valued functions. These functions are called Lyapunov matrices. The theory of the matrices is a rather young topic. Therefore, principal results with respect to the existence and uniqueness of the matrices are included. Some important applications of the functionals are pointed out. A brief historical survey ends the contribution.

1. Introduction

In this contribution we give an account of the present state of art in the area of quadratic functionals with prescribed time derivatives.

There are several issues that will be addressed in this paper, in particular we focus on the structure of the functionals, we are interested also in lower and upper bounds for them. The functionals are defined by special matrix valued functions known as Lyapunov matrices. The matrices are as important for the functionals, as the classical Lyapunov matrices are for the quadratic Lyapunov functions in the case of delay free systems. This explains the reason why we dedicate so much attention to existence and computation of the matrices. By definition Lyapunov matrices are solutions of a matrix delay equation which satisfy two additional properties. The delay matrix equation along with the properties is a counterpart of the classical Lyapunov matrix equation.

One of our specific goals is to demonstrate that the computed quadratic functionals can be effectively used in the stability, and robust stability analysis of time delay systems.

We begin our exposition with the case of single delay systems in the next section. Section 3 is dedicated to the case of systems with several delays. In Section 4 we address the case of systems with distributed delay.

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2. Single delay systems

We begin with the class of single delay systems. There are several reasons for restricting attention to this particular class before proceeding to more general ones. First, from the methodological point of view, it seems that dealing with single delay systems simplifies the understanding of basic concepts, and creates a firm basis for developing of the concepts for more general cases. Second, for single delay system, we often obtain more complete results than in a more general setting. And finally, results for the single delay case are not so cumbersome as that for the more general classes of time-delay systems.

2.1. System description

We consider a system of the form

$$\frac{dx(t)}{dt} = A_0x(t) + A_1x(t-h), \quad t \geq 0. \tag{1}$$

Here A_0, A_1 are given $n \times n$ matrices, and $h > 0$ is the time delay.

Let $\varphi : [-h, 0] \rightarrow R^n$ be an initial function. We assume that function φ belongs to the space of piece-wise continuous functions, $PC([-h, 0], R^n)$. Let $x(t, \varphi)$ stand for the solution of system (1) with an initial function φ,

$$x(\theta, \varphi) = \varphi(\theta), \quad \theta \in [-h, 0],$$

and $x_t(\varphi)$ denotes the restriction of the solution to the segment $[t-h, t]$,

$$x_t(\varphi) : \theta \rightarrow x(t+\theta, \varphi), \quad \theta \in [-h, 0].$$

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In some cases, when the initial function is not important, or is well defined from the context, we will use notations $x(t)$ and x_t, instead of $x(t,\varphi)$ and $x_t(\varphi)$.

The euclidean norm will be used for vectors, and the induced matrix norm for matrices. For the elements of the space $PC([-h,0],R^n)$ we will use the uniform norm

$$||\varphi||_h = \sup_{\theta\in[-h,0]} ||\varphi(\theta)||.$$

2.2. Exponential stability

Here we introduce the stability concept that will be used in the rest of the contribution.

Definition 1. (Bellman & Cooke, 1963) We say that a time delay system is exponentially stable if there exist $\gamma \geq 1$ and $\sigma > 0$, such that any solution $x(t,\varphi)$ of the system satisfies the inequality

$$||x(t,\varphi)|| \leq \gamma e^{-\sigma t}||\varphi||_h, \quad t \geq 0.$$

The following statement is a simplified version of the classical Krasovskii theorem. It provides sufficient conditions for the exponential stability of system (1).

Theorem 1. (Krasovskii, 1956) *System (1) is exponentially stable if there exists a functional*

$$v : PC([-h,0],R^n) \to R,$$

such that the following conditions hold:

1. *For some positive* α_1, α_2

$$\alpha_1||\varphi(0)||^2 \leq v(\varphi) \leq \alpha_2||\varphi||_h^2.$$

2. *For some* $\beta > 0$ *the inequality*

$$\frac{d}{dt}v(x_t) \leq -\beta||x(t)||^2, \quad t \geq 0,$$

holds along the solutions of the system.

2.3. Problem formulation

Motivated by Theorem 1 we address quadratic functionals that satisfy the theorem conditions.

There are two different possibilities to derive such functionals. In the first one we select a particular functional that satisfies the first condition of the theorem, and then check the second condition of the theorem. Usually results obtained in this way are given in the form of special LMIs. In Richard (2003) one can find a detailed survey of recent results in this direction.

In the other option, following one of the basic ideas of the Lyapunov method, we select first a time derivative, and then compute a functional, which time derivative along the solution of system (1) coincides with the selected one. In this contribution we do not treat the first possibility, but concentrate on the second one.

Since system (1) is linear and time-invariant, it seems natural to start with the case when the selected time derivative is a quadratic form.

Problem 1. Given a symmetric matrix W, we are looking for a functional

$$v_0 : PC([-h,0],R^n) \to R,$$

such that along the solutions of system (1) the following equality holds

$$\frac{d}{dt}v_0(x_t) = -x^T(t)Wx(t), \quad t \geq 0.$$

2.4. Lyapunov matrices and functionals

In this subsection we present a solution of Problem 1. But first we introduce Lyapunov matrices for system (1).

Definition 2. We say that $n \times n$ matrix $U(\tau)$, $\tau \in [-h,h]$, is a Lyapunov matrix of system (1) associated with a symmetric matrix W if it satisfies the properties

1. Dynamic property:

$$\frac{dU(\tau)}{d\tau} = U(\tau)A_0 + U(\tau-h)A_1, \quad \tau \in [0,h]. \tag{2}$$

2. Symmetry property:

$$U(-\tau) = U^T(\tau), \quad \tau \in [0,h]. \tag{3}$$

3. Algebraic property:

$$U(0)A_0 + U(-h)A_1 + A_0^TU(0) + A_1^TU(h) = -W. \tag{4}$$

Theorem 2. (Infante & Castelan, 1978; Repin, 1965) *Let* $U(\tau)$ *be a Lyapunov matrix associated with a symmetric matrix W. The functional*

$$\begin{aligned} v_0(\varphi) = {} & \varphi^T(0)U(0)\varphi(0) + 2\varphi^T(0)\int_{-h}^{0} U(-h-\theta)A_1\varphi(\theta)d\theta \\ & + \int_{-h}^{0}\varphi^T(\theta_1)A_1^T\left[\int_{-h}^{0}U(\theta_1-\theta_2)A_1\varphi(\theta_2)d\theta_2\right]d\theta_1 \end{aligned} \tag{5}$$

solves Problem 1.

2.5. Lyapunov matrices: Existence and uniqueness issues

Definition 2 raises the following question: When does Lyapunov matrix exist? Here we give a detailed account of the existence and uniqueness issues.

We begin with the definition of an important condition for system (1). For the delay free case this condition is well-known, and guarantees that the classical Lyapunov matrix equation admits a unique solution for any choice of matrix W.

Definition 3. We say that system (1) satisfies Lyapunov condition if the spectrum of the system,

$$\Lambda = \{s | det\,(sI - A_0 - e^{-sh}A_1) = 0\},$$

does not contain a point s_0, such that $-s_0$ also belongs to the spectrum.

Remark 1. If system (1) satisfies the Lyapunov condition, then it has no eigenvalues on the imaginary axis of the complex plane.

Computation of a Lyapunov matrix can be reduced to a special boundary value problem for an auxiliary system of delay free linear matrix differential equations. In order to demonstrate this we introduce two auxiliary matrices

$$Y(\tau) = U(\tau), \quad Z(\tau) = U(\tau-h), \quad \tau \in [0,h]. \tag{6}$$