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Systematic and effective design of nonlinear feedback controllers via the state-dependent Riccati equation (SDRE) method☆

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ABSTRACT

Since the 1990s, state-dependent Riccati equation (SDRE) strategies have emerged as general design methods that provide a systematic and effective means of designing nonlinear controllers, observers and filters. These methods overcome many of the difficulties and shortcomings of existing methodologies, and deliver computationally simple algorithms that have been highly effective in a variety of practical and meaningful applications in very diverse fields of study. These include *missiles*, *aircraft*, *unmanned aerial vehicles*, *satellites* and *spacecraft*, *ships*, *autonomous underwater vehicles*, *automotive systems*, *biomedical systems*, *process control*, and *robotics*, along with *various benchmark problems*, as well as nonlinear systems exhibiting several interesting phenomena such as *parasitic effects* of friction and backlash, *unstable nonminimum-phase dynamics*, *time-delay*, *vibration* and *chaotic behavior*. SDRE controllers, in particular, have become very popular within the control community, providing attractive *stability*, *optimality*, *robustness* and *computational* properties, making *real-time implementation in feedback form* feasible. However, despite a documented history of SDRE control in the literature, there is a significant lack of theoretical justification for logical choices of the design matrices, which have depended on intuitive rules of thumb and extensive simulation for evaluation and performance. In this paper, the capabilities and design flexibility of SDRE control are emphasized, addressing the issues on systematic selection of the design matrices and going into detail concerning the art of systematically carrying out an effective SDRE design for systems that both do and do not conform to the basic structure and conditions required by the method. Several situations that prevent the direct application of the SDRE technique, such as the presence of control and state constraints, are addressed, demonstrating how these situations can be readily handled using the method. In order to provide a clear understanding of the proposed methods, systematic and effective design tools of SDRE control are illustrated on a single-inverted pendulum nonlinear benchmark problem and a practical application problem of optimally administering chemotherapy in cancer treatment. Lastly, real-time implementation aspects are discussed with relevance to practical applicability.

1. Introduction

After several decades of enlightening research, numerous techniques currently exist for the synthesis of control laws for nonlinear systems. In particular, the development of sophisticated and rigorous mathematical framework for the formulation and analysis of procedures for systematic design of nonlinear controllers has attracted significant research interest since the 1970s (for example, see Isidori, 1995; Khalil, 2002; Krstic, Kanellakopoulos, & Kokotović, 1995; Slotine & Li, 1991; van der Schaft, 1999 and the references therein). Despite recent advances, however, there remain many unsolved problems, so much so that control practitioners often complain about the inapplicability of contemporary theories to *realistic* control design problems. Although several methods are fairly well established theoretically, there is a lack of a unified control methodology that, in addition to *stability*, has the ability to address *performance* and *robustness* properties to a satisfying extent for a wide class of nonlinear systems. In fact, most of the techniques developed have very limited applicability because of the strong conditions imposed on the system. More importantly, however, the resulting control law may not have a very intuitive design orientation for practical implementation, because additional requirements such as state and input specifications must also be met in practice, so that the design must accommodate limitations on the allowable states and available control inputs. Control system designers continue to

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strive for control algorithms that are systematic, simple to implement, and yet optimize performance, providing tradeoffs between control effort and allowable state errors.

State-dependent Riccati equation (*SDRE*) strategy (sometimes referred to as the *frozen Riccati equation*) has emerged as a very attractive tool for the systematic design of nonlinear controllers, and has become very popular within the control community over the last decade, providing an extremely effective algorithm for synthesizing nonlinear feedback controls by allowing nonlinearities in the system states while additionally offering great design flexibility through design matrices. This method, originally proposed by Pearson (1962) and later expanded by Wernli and Cook (1975), was independently studied by Cloutier, D'Souza, and Mracek (1996) and Mracek and Cloutier (1998), and alluded to by Friedland (1996). The method entails factorization (that is, *parameterization*) of the nonlinear dynamics into the product of a matrix-valued function (which depends on the state) and the state vector. In so doing, the SDRE algorithm fully captures the nonlinearities of the system, bringing the nonlinear system to a (*nonunique*) linear structure having *state-dependent coefficient* (SDC) matrices, and minimizing a nonquadratic performance index having a quadratic-like structure. The non-uniqueness of the SDC parameterization creates extra degrees of freedom that can be used to enhance controller performance. An *algebraic Riccati equation* (ARE) using the SDC matrices is then solved online to give the suboptimum (and in some cases optimum) control law. The coefficients of this ARE vary with the given point in state space. The algorithm thus involves solving, at a given point in state space, an algebraic *state-dependent Riccati equation*, or *SDRE*, whose pointwise stabilizing solution during state evolution yields the SDRE nonlinear feedback control law. As the SDRE depends only on the current state, the computation can be carried out online, in which case the SDRE is defined along the state trajectory. This is obviously desirable and makes implementation in real time in closed-loop form (using state feedback) feasible.

From a computational standpoint, SDRE control, based on nonlinear parameterization, provides a numerically efficient method that involves only an ARE, where application to a very general set of problems is considered by retaining a feedback control structure. This is an appealing alternative to tedious tasks involved with solving nonlinear *two-point boundary value problems* or *Hamilton-Jacobi-Bellman partial differential equations* associated with nonlinear optimal control problems. In an attempt to bring linear-quadratic (LQ) insight to nonlinear systems, the problem formulation allows for a tradeoff between control effort and state regulation, which is readily transparent in the nonlinear scheme. Therefore, as in the LQ setting, the state and control signal magnitudes may be easily and transparently tuned by adjusting the entries in the penalty matrices. As such, the technique offers the same type of design tradeoff flexibility as with existing linear H_2 control methods and, in particular, shares several symmetries with the commonly used *linear-quadratic regulator* (LQR), making it easily accessible to most control system designers.

The theory developed to date on *existence of solutions* as well as *stability and optimality properties* associated with SDRE controllers for *SDRE nonlinear regulation* have been presented in a recent survey in Çimen (2008). This paper now focuses on the capabilities, design flexibility and art of systematically carrying out an effective SDRE design for systems that both do and do not conform to the basic structure and conditions required by the method, demonstrating how several situations that prevent the direct application of the SDRE technique, such as the presence of control and state constraints, can be readily handled using this method. Real-time implementation aspects are also considered with relevance to practical applicability. While this paper is centered on the SDRE nonlinear regulator, the techniques presented here for producing effective SDRE designs and for handling systems that do not conform to the basic structure and conditions required for the direct application of the method apply to all of the various SDRE techniques (see Cloutier, 1997), which are defined by their linear-like structures with SDC matrices.

The remainder of this paper is organized as follows: In Section 2, the *SDRE method* is first reviewed, presenting the formulation of the optimal control problem for *nonlinear regulation*, with a brief overview of the concept of *extended linearization*, the *SDRE controller structure and conditions*, and the *additional degrees of freedom* provided by the nonuniqueness of the SDC parameterization. Sections 3 presents an overview of the capabilities of SDRE control, assessing the *design flexibility* of the method afforded via the additional degrees of freedom of the SDC parameterization and the state-dependent weightings, and addressing the issues on systematic selection of these design matrices for carrying out an effective SDRE design. The SDRE nonlinear regulator with *integral servomechanism* action is then discussed for *tracking* or *command following*, followed by SDRE controller design with *partial-state feedback*, as opposed to full-state information. An overview concerning the art of carrying out an effective and systematic SDRE design for *nonconforming systems* is presented in Section 4, demonstrating how numerous systems that do not meet the basic structure and conditions required for the direct application of the SDRE technique can be systematically converted to systems having the proper structure and conditions. In order to illustrate the application and validity of the proposed methods, two nonlinear *applications* (a benchmark problem and a practical application problem) are presented in Section 5. *Real-time implementation* aspects of SDRE nonlinear feedback control synthesis are subsequently conveyed in Section 6. Finally, the paper is concluded with a summary in Section 7.

The symbolic notation adhered to in this paper is the standard in conventional control theory. $\mathbb{R}$ is the set of real numbers. For any positive integer n, $\mathbb{R}^n$ is n-dimensional real Euclidean space. At an intermediate value of time $t \in [0, \infty)$, the state vector is denoted by $\mathbf{x}(t) = [x_1(t), \ \ldots, \ x_n(t)]^T \in \mathbb{R}^n$ and the control (input) vector by $\mathbf{u}(t) = [u_1(t), \ \ldots, \ u_m(t)]^T \in \mathbb{R}^m$ $(m \geq 1)$. The state set, represented by Ω, is a bounded open subset of $\mathbb{R}^n$ Euclidean space that contains the origin, such that $\mathbf{0} \in \Omega \subseteq \mathbb{R}^n$. A vector field $\mathbf{f} : \Omega \to \mathbb{R}^n$ is an n-dimensional *column vector*. $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices. The transpose of some matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ is denoted by $\mathbf{M}^T$. The set of eigenvalues of a square matrix $\mathbf{M}$ is denoted by $\lambda_i(\mathbf{M})$. $\mathbf{P} > \mathbf{0}$ $(\mathbf{P} \geq \mathbf{0})$ for some matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$ is used to mean that the matrix is positive-(semi)definite. A function is said to be of class $C^k(\Omega)$ (or simply C^k) if it is continuously differentiable k times in Ω. In this regard, $C^0(\Omega)$ (or C^0) stands for the class of continuous functions in Ω.

2. SDRE method

2.1. Nonlinear optimal regulation

Consider the continuous-time, deterministic, full-state feedback, infinite-time horizon nonlinear optimal regulation (stabilization) problem, where the system is autonomous, nonlinear in the state, and affine (linear) in the input, having dynamics

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{1}$$

with state vector $\mathbf{x} \in \mathbb{R}^n$ and (unconstrained) input vector $\mathbf{u} \in \mathbb{R}^m$, such that $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{B} : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, with $\mathbf{B}(\mathbf{x}) \neq \mathbf{0}$ $\forall \mathbf{x}$. In this context, the minimization of an *infinite-time* performance criterion with a *convex* integrand, *nonquadratic* in $\mathbf{x}$ but *quadratic* in $\mathbf{u}$, is considered, given by

$$J(\mathbf{x}_0, \ \mathbf{u}(\cdot)) = \frac{1}{2}\int_0^\infty \left\{\mathbf{x}^T(t)\mathbf{Q}(\mathbf{x})\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}(\mathbf{x})\mathbf{u}(t)\right\} \, dt. \tag{2}$$