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On the optimal parameter of the composite Laplacian quadratics function[☆]

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ABSTRACT

Recently, the composite quadratic Lyapunov function has been extended to study of multi-agent systems, leading to the so-called composite Laplacian quadratics (CLQs) function. Compared with quadratic Lyapunov functions, the CLQs function can yield a larger convergence region and is particularly useful in stabilization of multi-agent systems with complex dynamics, such as differential inclusions. In the definition of the CLQs function, an optimal vector parameter plays a critical role in determining the value of the CLQs function and in constructing stabilization laws derived from the CLQs function. This paper focuses on the properties of the optimal parameter of the CLQs function. The uniqueness of the optimal parameter is established. A distributed computation approach is further proposed, which is useful in computing the optimal parameter. The robustness issue of the optimal parameter is also investigated for a multi-agent system described by linear differential inclusions. Finally, a numerical example is provided to validate the proposed theoretical results.

1. Introduction

Recently, the function of composite Laplacian quadratics is proposed for synthesis and analysis of multi-agent systems (Chen, Xiang, & Ren, 2015), which can be seen as an extension of the composite quadratic function (Goebel, Teel, Hu, & Lin, 2006; Hu & Lin, 2004) to the study of multi-agent systems, where the Laplacian matrix is introduced to describe the communication graph among agents. The function is particularly useful in stabilization of multi-agent systems whose dynamics are governed by linear differential inclusions (LDIs), which can be used to model a wide class of practical systems (Boyd, Ghaoui, Feron, & Balakrishnan, 1994). It has been demonstrated that the CLQs function can yield a larger stabilization region, as compared to that given by quadratic Lyapunov functions (Hu, 2007). In addition, by the technique of global linearization in Liu (1968) and Liu, Saeks, and Leake (1971), a nonlinear time-varying system can be transformed into an LDI system, which suggests that the CLQs function can be employed to stabilize nonlinear time-varying systems as well (Boyd et al., 1994). The function is used to design a nonlinear consensus algorithm for a multi-agent system (Chen et al., 2015), for which consensus cannot be reached via linear algorithms designed under the quadratic stabilization framework. Because consensus is the basis for more involved cooperative control problems, the CLQs function has potential applications in other cooperative control problems of multi-agent systems such as distributed optimization (Nedic & Ozdaglar, 2009; Sarlette & Sepulchre, 2009).

Construction of Lyapunov functions is one of the central tasks in study of multi-agent systems. Lyapunov functions are not only used for analysis of multi-agent systems, but also play a critical rule in controller synthesis of multi-agent systems. One type of quadratic functions that is commonly used is $V(x) \triangleq x^T(L \otimes P)x$, where L is the Laplacian matrix of an undirected graph and P is a positive definite matrix (Olfati-Saber & Murray, 2004; Ren & Beard, 2008; Ren, Beard, & Atkins, 2007). Quadratic Lyapunov functions can also be designed for multi-agent systems under directed network topologies and various input or communication constrains (Meng & Lin, 2014; Meng, Zhao, & Lin, 2013). Although quadratic Lyapunov functions serve as a fundamental tool in stability analysis and control synthesis, their limitations have been revealed in some papers (Moreau, 2005; Ooba, 2003). Therefore,

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some non-quadratic Lyapunov functions have been proposed as a complement of quadratic Lyapunov functions. The set-valued Lyapunov function

$V_{\text{conv}} \triangleq (\text{conv}\{x_1, \ldots, x_n\})^n$,

is used in Moreau (2005) to analyze the convergence of a nonlinear multi-agent model, where $\text{conv}\{x_1, \ldots, x_n\}$ denotes the convex hull and $(\cdot)^n$ denotes the Cartesian product. Different from the quadratic Lyapunov function, the set-valued Lyapunov function can be used to conclude that the state of the system converges to one equilibrium out of a continuum of equilibria. Finally, it is worth mentioning that other types of Lyapunov functions such as the max–min function $V_{\max,\min} \triangleq \max\{x_1, \ldots, x_n\} - \min\{x_1, \ldots, x_n\}$ (Cao, Ren, & Meng, 2010; Moreau, 2004) are also employed in the literature for multi-agent systems.

As an extension of the composite quadratic function, the CLQs function inherits a very good property of the composite quadratic function, that is, it is continuously differentiable. This property is particularly useful in constructing continuous control laws from the CLQs. Note that a continuous control law, if exists, is always preferable to a discontinuous control law due to various considerations, such as avoiding the chattering effect. The value of the CLQs function is normally determined by solving an optimization problem, whose solution is given in terms of a vector parameter. The optimal value of the vector parameter plays a critical role not only in determining the value of the CLQs function but also in designing the control law under the CLQs function. The challenge of determining the optimal parameter is that the parameter is time-varying and depends on the state of the system. This implies that global information of the multi-agent system is needed in determining the optimal parameter. This motivates the problem of designing a distributed algorithm for determining the optimal parameter. Moreover, the computation of the optimal parameter needs time, which is not negligible for multi-agent systems where communication is involved. One possible way to solve the computational time issue is to use "out-dated" values of the optimal parameter. The question is what is the upper bound on the deviation of the "out-dated" values from the true value of the optimal parameter that can be employed while guaranteeing the stability of the resulting system. This motivates the robustness issue of the optimal parameter of the CLQs function in controller synthesis of multi-agent systems.

The contributions of this paper are stated as follows. First, the uniqueness of the optimal parameter of the CLQs function is established. Second, a distributed algorithm is proposed which plays an important role in computing the optimal parameter of the CLQs function and is necessary for the distributed implementation of the control algorithm designed via the CLQs function. Finally, the robustness issue of the optimal parameter is investigated for a multi-agent system described by LDIs. It is proved that if the calculation error of the optimal parameter is smaller than certain upper bound, then consensus of the resulting controlled multi-agent system can still be guaranteed even if the algorithm employs an "inaccurate" value of the optimal parameter.

The rest of the paper is organized as follows. In Section 2, some notations and mathematical preliminaries are introduced. The uniqueness result of the optimal parameter of the CLQs function is established in Section 3. Section 4 presents a distributed algorithm for computing the optimal parameter of the CLQs function. The robustness issue of the optimal parameter is discussed in Section 5. A simulation example is then given in Section 6 to verify the obtained result on the robustness of the optimal parameter. Finally, Section 7 concludes this paper and gives some future directions.

2. Preliminaries

Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{R}^+$ the set of all positive real numbers. Let $\mathbb{R}^n$ denote the set of n-dimensional real vectors and $\mathbb{R}^{m\times n}$ the set of $m \times n$ real matrices. Let $I_n \in \mathbb{R}^{n\times n}$ be the n-dimensional identity matrix, $\mathbf{0}_n \in \mathbb{R}^n$ the vector with all zeros, and $\mathbf{1}_n \in \mathbb{R}^n$ the vector with all ones. Let $\mathrm{i}_j \in \mathbb{R}^n$ denote the vector with all zeros, except that the jth entry is one. The subscripts of I_n, $\mathbf{0}_n$, and $\mathbf{1}_n$ might be dropped if no confusion arises in the context. For a vector $x \in \mathbb{R}^n$, define $\|x\| \triangleq (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$ and let $\text{diag}(x) \in \mathbb{R}^{n\times n}$ be the diagonal matrix constructed from x with the elements in the main diagonal being the elements of x. For a matrix $A \in \mathbb{R}^{m\times n}$, $\|A\| \triangleq \sqrt{\rho\left(AA^T\right)}$ denotes its *induced two-norm*, where A^T is the transpose and $\rho(\cdot)$ is the spectral radius. The term "if and only if" is abbreviated as "iff". Let S be a compact convex set. A point $x_0 \in S$ is an *extreme point* of S if it cannot be represented as the convex combination of other points in S. A hyperplane $c^T x = k$ is a *supporting hyperplane* of S at $x_0 \in \partial S$ if $c^T x \le k$ for all $x \in S$ and $c^T x_0 = k$, where ∂S denotes the boundary of S.

A *graph* is defined as $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the set of edges. A graph is *simple* if it does not contain self-loops, nor have multiple edges between two nodes. Graph $\mathcal{G}$ is *undirected* if, for all $u, v \in \mathcal{V}$, $(u, v) \in \mathcal{E} \Longleftrightarrow (v, u) \in \mathcal{E}$. In this paper, only simple and undirected graphs are considered. The *order* and *size* of $\mathcal{G}$ are denoted, respectively, by $n \triangleq |\mathcal{V}|$ and $m \triangleq |\mathcal{E}|$, where $|\cdot|$ denotes the number of elements in a set. A *path* from node v_1 to node v_k is a sequence of nodes $v_1, \ldots, v_k$, such that for each i, $1 \le i \le k-1$, (v_i, v_{i+1}) is an edge. A graph is *connected* if there is a path from any node to any other node in the graph. Let $A \triangleq [a_{ij}] \in \mathbb{R}^{n\times n}$ be the *adjacency matrix* of $\mathcal{G}$. The *degree* of node i is defined as $d_i \triangleq \sum_{j=1}^n a_{ij}$, and the *degree matrix* is defined as $D \triangleq \text{diag}([d_1, \ldots, d_n]) \in \mathbb{R}^{n\times n}$. The *Laplacian matrix* of $\mathcal{G}$ is then given by $L \triangleq D - A \in \mathbb{R}^{n\times n}$. It can be verified that the Laplacian matrix L is positive semi-definite, and has a zero eigenvalue whose normalized eigenvector is $(1/\sqrt{n})\mathbf{1}_n$. Without loss of generality, let $0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ denote the n eigenvalues of L. Here, the second-smallest eigenvalue λ_2 is called the *algebraic connectivity* of graph $\mathcal{G}$, which is related to the connectivity of graph $\mathcal{G}$ by the following lemma.

Lemma 1 (Godsil & Royle, 2001)**.** *The algebraic connectivity $\lambda_2 > 0$ iff $\mathcal{G}$ is a connected graph.*

For a symmetric block matrix $\mathfrak{X} = \begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{B}^T & \mathfrak{C} \end{bmatrix}$, its Schur complement is defined by $\mathfrak{A} - \mathfrak{B}\mathfrak{C}^{-1}\mathfrak{B}^T$ if $\mathfrak{C}^{-1}$ exists. The positive definiteness (positive semi-definiteness) of the block matrix $\mathfrak{X}$ can be verified via its Schur complement.

Lemma 2 (Schur Complement, Boyd et al., 1994)**.** *For the symmetric matrix $\mathfrak{X}$, the following holds*

- $\mathfrak{X} > 0 \Longleftrightarrow \mathfrak{C} > 0, \mathfrak{A} - \mathfrak{B}\mathfrak{C}^{-1}\mathfrak{B}^T > 0$;
- $\mathfrak{X} \ge 0 \Longleftrightarrow \mathfrak{C} > 0, \mathfrak{A} - \mathfrak{B}\mathfrak{C}^{-1}\mathfrak{B}^T \ge 0$.

Let $\mathfrak{A}(\mathfrak{x})$ be an invertible matrix depending on a real parameter $\mathfrak{x} \in \mathcal{X} \subset \mathbb{R}$. The following lemma shows how to calculate the derivative of the inverse matrix $\mathfrak{A}^{-1}$.

Lemma 3 (Horn & Johnson, 2012)**.** *Let B be a matrix such that $\|B\| < 1$, then the matrix $(I + B)$ is invertible, and $\|(I + B)^{-1}\| \le \frac{1}{1-\|B\|}$.*