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Stability of discrete-time switching systems with constrained switching sequences[☆]

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ABSTRACT

We introduce a novel framework for the stability analysis of discrete-time linear switching systems with switching sequences constrained by an automaton. The key element of the framework is the algebraic concept of multinorm, which associates a different norm per node of the automaton, and allows to exactly characterize stability. Building upon this tool, we develop the first arbitrarily accurate approximation schemes for estimating the *constrained* joint spectral radius $\hat{\rho}$, that is the exponential growth rate of a switching system with constrained switching sequences. More precisely, given a relative accuracy $r > 0$, the algorithms compute an estimate of $\hat{\rho}$ within the range $[\hat{\rho}, (1+r)\hat{\rho}]$. These algorithms amount to solve a well defined convex optimization program with known time-complexity, and whose size depends on the desired relative accuracy $r > 0$.

1. Introduction

In this paper, we study discrete-time linear switching systems having the particularity that their switching sequences are constrained by logical rules. We begin with an example introducing such systems.

Given an *unstable* matrix $A_1 \in \mathbb{R}^{n\times n}$ and an input-to-state matrix $B \in \mathbb{R}^{n\times m}$, one computes a control gain matrix $K \in \mathbb{R}^{m\times n}$ such that $A_2 = (A_1 + BK)$ is stable. The matrix A_2 dictates the closed-loop dynamics of a plant, $x_{t+1} = A_2 x_t$, whose stability is ensured by a state-feedback controller. Let us now consider that the controller can fail at any time t, such that the dynamics at that time are given by $x_{t+1} = A_1 x_t$. Then, the dynamics of the plant with failures can be modelled as a switching system

$$x_{t+1} = A_{\sigma(t)} x_t,$$

where $\sigma(t) \in \{1, 2\}$ is the *mode* of the system and $\sigma(0), \sigma(1), \ldots,$ is the *switching sequence* that drives the system. Without more information on the occurrences of the failures, we can only assume that the system is unstable. Indeed, in the case of a permanent failure, represented by the switching sequence $\sigma(t) = 1, \forall t \geq 0$, the plant would follow the unstable dynamics $x_{t+1} = A_1 x_t$ at every time $t \geq 0$. However, if we knew with *certainty* that the failure cannot occur *more than twice* in a row, then the above switching sequence would no longer be possible, and the system could very well be stable.

This paper provides tools for the stability analysis of switching systems with constrained switching sequences, as in the example above. We say that the switching system on the matrix set $\mathbf{\Sigma} = \{A_1, A_2, \ldots, A_N\}$ is stable if and only if, for all *accepted* switching sequences $\sigma(0), \sigma(1), \ldots,$ we have $\lim_{t\to\infty} A_{\sigma(t)} \cdots A_{\sigma(0)} = 0$.

Switching systems find applications in many theoretical and engineering related domains (Hernandez-Vargas, Middleton, & Colaneri, 2011; Jungers, 2009; Jungers, D'Innocenzo, & Di Benedetto, 2012; Jungers & Heemels, 0000; Liberzon & Morse, 1999; Olfati-Saber & Murray, 2004), and the stability of switching systems is known to be a challenging question (Liberzon & Morse,

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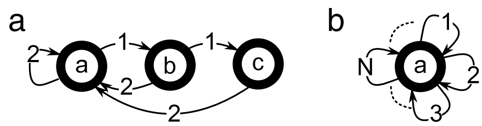

Fig. 1. The labels are represented on the edges. Fig. 1(a) corresponds to the example in the introduction, where mode "1" cannot occur more than twice in a row. Node "a" is reached when the controller works, "b" is reached after one failure, and "c" after two failures. The automaton of Fig. 1(b) accepts arbitrary switching sequences on N modes.

1999; Lin & Antsaklis, 2009; Shorten, Wirth, Mason, Wulff, & King, 2007).

If one does not impose any constraint on switching sequences, the resulting system is called an *arbitrary* switching system. These systems have received a lot of attention in the past (e.g., Agrachev & Liberzon, 2001, Ahmadi, Jungers, Parrilo, & Roozbehani, 2014, Ando & Shih, 1998 and Jungers, 2009). The stability of an arbitrary switching system on a set of matrices $\mathbf{\Sigma}$ is characterized by its *joint spectral radius* (JSR) $\hat{\rho}(\mathbf{\Sigma})$ (introduced in Rota & Strang, 1960). It represents the worst case exponential growth rate of the system, and stability is equivalent to $\hat{\rho}(\mathbf{\Sigma}) < 1$, which is also equivalent to exponential stability. There has been a lot of research effort towards the computation and approximation of the JSR (see e.g. Ahmadi et al., 2014, Blondel, Nesterov, & Theys, 2005, Jungers, 2009, Vankeerberghen, Hendrickx, & Jungers, 2014 and references therein). One common way to do so is by computing a contractive invariant norm for the system (Ando & Shih, 1998; Athanasopoulos & Lazar, 2014; Blondel et al., 2005; Parrilo & Jadbabaie, 2008), which *always exists* for stable arbitrary switching systems. For any level of relative accuracy $r > 0$, one can approximate these norms with quadratic/sum-of-square polynomials (Blondel et al., 2005; Parrilo & Jadbabaie, 2008) and provide an *upper bound* on the joint spectral radius within the range $[\hat{\rho}, (1+r)\hat{\rho}]$. The computation of this estimate is done with finite time-complexity.

Our focus is on the stability of switching systems having logical rules on their switching sequences, such as the ones studied in Ahmadi et al. (2014), Bliman and Ferrari-Trecate (2003), Dai (2012), Essick, Lee, and Dullerud (2014), Kozyakin (2014), Kundu and Chatterjee (2015), Lee and Dullerud (2006a,b), Lin and Antsaklis (2009), Philippe and Jungers (2015a) and Wang, Roohi, Dullerud, and Viswanathan (2014). We refer to these as *constrained switching systems*, and represent the rules by using an *automaton*. An automaton is a strongly connected, directed and labelled graph $\mathbf{G}(V, E)$, with N_V nodes in V and N_E edges in E. The edge $(v, w, \sigma) \in E$ between the two nodes $v, w \in V$ carries the *label* $\sigma \in \{1, \dots, N\}$, which maps to a mode of the switching system. A sequence of modes $\sigma(0), \sigma(1), \dots$, is *accepted* by the graph $\mathbf{G}$ if there is a path in $\mathbf{G}$ carrying the sequence as the succession of the labels on its edges. We do not specify an initial and final node for accepted paths, in that we depart from the usual definition for an automaton (see Lothaire, 2002, Section 1.3). The accepted switching sequences form a symbolic dynamical system called *sofic shift* (see Lothaire, 2002, Section 1.5). Examples of automata are given in Fig. 1.

The system on the automaton $\mathbf{G}$ with matrix set $\mathbf{\Sigma}$ is denoted $S(\mathbf{G}, \mathbf{\Sigma})$. The stability of $S(\mathbf{G}, \mathbf{\Sigma})$ is characterized by the *constrained joint spectral radius*, introduced by Dai (2012). A proof of the following is given in the Appendix.

Theorem 1.1 (*Dai, 2012, Corollary 2.8*). *A constrained switching system $S(\mathbf{G}, \mathbf{\Sigma})$ is stable if and only if its* constrained joint spectral radius *(CJSR), defined as*

$$\hat{\rho}(S) \triangleq \lim_{t\to\infty} \max_{\sigma(\cdot)} \{\|A_{\sigma(t-1)} \cdots A_{\sigma(0)}\|^{1/t} : \\ \sigma(0), \dots, \sigma(t-1) \text{ is accepted by } \mathbf{G}\}, \quad (1)$$

satisfies $\hat{\rho}(S) < 1$. This also implies exponential stability. For all accepted switching sequences,

$$\exists K \geq 1,\ 0 < \rho < 1 : \forall T \geq 0, \quad \|A_{\sigma(T-1)} \cdots A_{\sigma(0)}\| \leq K\rho^T.$$

The CJSR, defined as (1), is independent of the norm used and homogeneous in $\mathbf{\Sigma}$.

To the best of our knowledge, previous works on the stability of constrained switching systems have focused on establishing algorithmically checkable stability conditions, without studying their conservativeness. There is a particular interest in using multiple quadratic Lyapunov functions as stability certificates (Bliman & Ferrari-Trecate, 2003; Branicky, 1998; Essick et al., 2014; Lee & Dullerud, 2006b; Lee & Khargonekar, 2009; Lin & Antsaklis, 2009). These approaches provide sets of LMIs whose feasibility is sufficient for stability. In Bliman and Ferrari-Trecate (2003), Lee and Dullerud (2006b), a hierarchy of more and more complex LMIs is presented such that, for any stable system, all LMIs starting from a certain level of complexity (depending on the system) are feasible. The methods discussed above can be used to obtain *upper bounds* on the CJSR (a feasible LMI indicates $\hat{\rho}(S) < 1$). However, accuracy guarantees on these bounds, similar to that existing on the JSR estimation, have not been proven yet.

The framework we introduce allows to obtain accuracy guarantees.[2] A direct approach could rely on building an arbitrary switching system whose JSR equals the CJSR of the constrained system (Kozyakin, 2014; Wang et al., 2014). We provide more efficient and intuitive techniques. We generalize the recent results from Ahmadi et al. (2014) towards constrained switching systems. In Ahmadi et al. (2014), the authors focus on systems $S(\mathbf{G}, \mathbf{\Sigma})$ with $\mathbf{G}$ accepting arbitrary switching sequences, and provide accuracy bounds for the JSR estimation using multiple Lyapunov functions. The generalization of these results to general constrained switching systems was left as an open question.

The plan of the paper is as follows. Section 2 introduces the algebraic concept of multinorm, which characterizes the stability of constrained switching systems as contractive norms do for arbitrary switching systems. In Section 3, we focus on the more algorithmic question of the approximation – in finite time and with arbitrary accuracy – of the CJSR of a system $S(\mathbf{G}, \mathbf{\Sigma})$. In Section 4, we illustrate our framework on a numerical example.

Notations. The matrix $A^\top \in \mathbb{R}^{m\times n}$ is the transpose of $A \in \mathbb{R}^{n\times m}$. A path p of length $T \geq 0$ in a graph $\mathbf{G}$ is a sequence of T consecutive edges. For a path p with length $T \geq 1$, by a slight abuse of notations, we let $A_p = A_{\sigma(T)} \cdots A_{\sigma(1)}$, where the $\sigma(1), \dots, \sigma(T)$ are the T labels along p. If $T = 0$, we let $A_p = I$, the identity matrix of $\mathbb{R}^n$.

2. Lyapunov functions for constrained switching systems

The stability of *arbitrary* switching systems is equivalent to the existence of a contractive norm serving as a Lyapunov function. We recall that a norm is a sub additive, positive definite and homogeneous function.

Proposition 2.1 (*E.g. Jungers, 2009, Proposition 1.4*). *The joint spectral radius of a set of matrices $\mathbf{\Sigma}$ is given by*

$$\hat{\rho}(\mathbf{\Sigma}) \triangleq \inf_{|\cdot|} \min_{\gamma} \left\{ \gamma : |Ax| \leq \gamma|x|,\ \forall x \in \mathbb{R}^n,\ A \in \mathbf{\Sigma} \right\}. \quad (2)$$

where the infimum is taken over all vector norms in $\mathbb{R}^n$.

A stable arbitrary switching system has $\hat{\rho}(\mathbf{\Sigma}) < 1$ (see Jungers, 2009) and from Proposition 2.1, there exists a norm $|\cdot|$ such that

[2] Preliminary results were presented in Philippe and Jungers (2015b).