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Brief paper

Weighted homogeneity and robustness of sliding mode control[☆]

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ABSTRACT

General features of finite-time-stable (FTS) homogeneous differential inclusions (DIs) are investigated in the context of sliding-mode control (SMC). The continuity features of the settling-time functions of FTS homogeneous DIs are considered, and the system asymptotic accuracy is calculated in the presence of disturbances, noises and delays. Performance of output-feedback multi-input multi-output homogeneous SMC systems is studied in the presence of relative degree fluctuations. The bifurcation of the kinematic-car-model relative degree is analyzed as an example.

1. Introduction

Conventional sliding mode (SM) control (SMC) (Edwards & Spurgeon, 1998; Utkin, 1992) is based on keeping $\sigma \equiv 0$ for an appropriate vector output σ called the sliding variable. It results in possibly dangerous high-frequency switching (chattering) (Boiko & Fridman, 2005; Fridman, 2003; Utkin, 1992). The relative degree of the components of σ should be 1, i.e. already $\dot{\sigma}$ should contain controls. Recall that the relative degree (Isidori, 1995) is roughly the lowest order of the output's total time derivative containing controls with non-zero coefficients.

High-order SMs (HOSMs) have overcome the relative degree restriction (Bartolini, Ferrara, & Usai, 1998; Bartolini, Pisano, Punta, & Usai, 2003; Boiko & Fridman, 2005; Levant, 1993; Plestan, Glumineau, & Laghrouche, 2008; Shtessel, Taleb, & Plestan, 2012). Introducing integrators, one also effectively attenuates the chattering.

The auxiliary dynamics of sliding variables is naturally described by differential inclusions (DIs). Finite-time (FT) stabilization of such DIs becomes the main SMC task. A control feedback yielding a FT stable (FTS) homogeneous DI solves the problem (Bacciotti & Rosier, 2005; Bernuau, Efimov, Perruquetti, & Polyakov, 2014; Bhat & Bernstein, 2000; Levant, 2003, 2005a; Orlov, 2005; Polyakov & Fridman, 2014). Respectively HOSM controllers impose homogeneous dynamics on the sliding variables. The lacking derivatives of σ are robustly estimated in FT by means of exact homogeneous differentiators (Levant, 2003). The error dynamics of a continuous-time system closed by discrete-time dynamics of an output-feedback controller can be considered as a special homogeneous hybrid dynamic system (Goebel, Sanfelice, & Teel, 2012; Goebel & Teel, 2010).

Hence, the homogeneity theory has become the main tool of SMC design, whereas the relative degree turns to be its main parameter. In particular, the theory provides estimations of the transient times and accuracies in the presence of disturbances (Bernuau, Efimov, Perruquetti, & Polyakov, 2014; Bhat & Bernstein, 2000; Goebel & Teel, 2010), and the asymptotic system accuracies in the presence of noises and time delays (Levant, 2005a).

Small dynamic uncertainties can lower the relative degree and destroy the above control design. Thus, the results (Levant, 2005a) are to be extended to such disturbed cases. It was proved in Bernuau, Efimov, and Perruquetti (2014); Bernuau, Efimov, Perruquetti, and Polyakov (2014), Goebel and Teel (2010) and Levant (2009) that in the presence of bounded disturbances homogeneous FTS DIs feature bounded FT attractors. Unfortunately these results do not consider time delays and sampling noises, and do not provide for the corresponding asymptotic accuracy estimations. This paper extends the results (Levant, 2005a) to disturbed FTS DIs and fills that gap.

The present paper studies some general features of FTS homogeneous DIs. In particular it corrects a few inaccuracies which appear in Levant (2005a) with respect to the continuity features of the settling-time functions, and extends and generalizes the

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accuracy estimations from Levant (2005a) and Livne and Levant (2014a). The asymptotics of the transient time and the accuracy of FTS homogeneous DIs in the presence of dynamic disturbances, sampling noises and time delays are calculated.

The results are applied to the analysis of disturbed multi-input multi-output (MIMO) systems under homogeneous output-feedback SMC. A case study considers the bifurcation of the kinematic-car-model relative degree. The asymptotic accuracies are calculated theoretically and confirmed by simulation.

Some notation and definitions

Let $s \in \mathbb{R}^m$, $\varpi \geq 1$. Denote $\|s\|_\varpi = (|s_1|^\varpi + \cdots + |s_m|^\varpi)^{1/\varpi}$, $\|s\|_\infty = \max\{|s_1|, \ldots, |s_m|\}$, $\|s\| = \|s\|_2$.

Any binary operation $\diamond$ of two sets is defined as $A \diamond B = \{a \diamond b \mid a \in A, b \in B\}$. A vector (point) is considered as a one-element set in that context. Let $s \in \mathbb{R}^m$, $A \subset \mathbb{R}^m$. Then the distance is defined, $\mathrm{dist}(s, A) = \inf\{|s - a| \mid a \in A\}$. A set-valued function $F(s)$ is called upper-semicontinuous if $\lim_{s \to \tilde{s}} \left[\sup\{\mathrm{dist}(z, F(\tilde{s}))|z \in F(s)\}\right] = 0$.

Denote $A^\varepsilon = \{s \in \mathbb{R}^m \mid \mathrm{dist}(s, A) \leq \varepsilon\}$. For any function F and set M denote $F(M) = \bigcup_{s \in M} F(s)$.

A scalar function $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^m$, is called *upper semicontinuous* (respectively, *lower semicontinuous*) at a point $s_0 \in D$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $f(s) \leq f(s_0) + \varepsilon$ (respectively, $f(s) \geq f(s_0) - \varepsilon$) for all $s \in D \cap \{s_0\}^\delta$.

2. Coordinate homogeneity and settling functions

Consider a Filippov DI

$$\dot{s} \in F(s), \quad s \in \mathbb{R}^m. \tag{1}$$

It means that $F(s) \subset \mathbb{R}^m$ is an upper-semicontinuous non-empty compact convex set-valued function (Filippov, 1988).

Such DIs feature the existence and extendability of local solutions, and their continuous dependence on initial conditions and the graph of the right-hand side (Filippov, 1988). Solutions of DI (1) are defined as locally absolutely continuous functions satisfying (1) for almost any t.

Let DI (1) be also homogeneous of the degree q. The latter means (Levant, 2005a) that $F(s) = \kappa^{-q} d_\kappa^{-1} F(d_\kappa s)$ for any $\kappa > 0$ with the homogeneity dilation

$$d_\kappa : (s_1, \ldots, s_m) \mapsto (\kappa^{w_1} s_1, \ldots, \kappa^{w_m} s_m), \quad w_1, w_2, \ldots, w_m > 0. \tag{2}$$

Here $w_i > 0$ are called the weights (homogeneity degrees) of s_i, $\deg s_i = w_i$. Denote $p = -q$. The homogeneity of DI (1) is equivalent to the invariance of (1) with respect to the time-coordinate transformation

$$G_\kappa : (t, s) \mapsto (\kappa^p t, d_\kappa s). \tag{3}$$

One can formally define $\deg t = p$.

Recall that a function $\phi(s)$ is called homogeneous with the homogeneity degree q, $\deg \phi = q$, if the identity $\phi(s) = \kappa^{-q} \phi(d_\kappa s)$ holds for all s and $\kappa > 0$. The standard definition (Bacciotti & Rosier, 2005) of the homogeneity of the differential equation $\dot{s} = f(s) = (f_1(s), \ldots, f_m(s))^T$ is that $\deg \dot{s}_i = \deg s_i - \deg t = \deg f_i$. Definitions coincide, if the equation $\dot{s} = f(s)$ is considered as the DI $\dot{s} \in \{f(s)\}$.

A *homogeneous norm* $\|s\|_h$ is any positive-definite continuous function of s of the weight 1. It is never smooth at 0, but $\|s\|_h = \left(|s_1|^{\varpi/w_1} + \cdots + |s_m|^{\varpi/w_m}\right)^{1/\varpi}$, $\varpi \geq \max_i w_i$, is 1-smooth at $s \neq 0$.

Note that all homogeneity degrees are simultaneously multiplied by $\lambda_w > 0$ as the result of the substitution $\kappa = \tilde{\kappa}^{\lambda_w}$. In particular, a non-zero homogeneity degree $q = -p$ can always be scaled to ± 1.

Proposition 1. *Let (1) be a Filippov homogeneous DI with the dilation (2) and the homogeneity transformation (3). Then for any $i = 1, \ldots, m$ either $w_i \geq p$, or the ith vector component of the inclusion is identical zero everywhere except the origin.*

Proof. Indeed, let $F(s)$ contain a vector $v = (v_1, \ldots, v_m)$ with $v_i \neq 0$. Thus $F(d_\kappa s) = \kappa^{-p} d_\kappa F(s)$ contains the vector $\kappa^{-p} d_\kappa v$ with its ith component equal to $\kappa^{w_i - p} v_i$. In the case $w_i < p$ this component tends to infinity for $\kappa \to 0$, and, respectively, due to the upper semicontinuity of F the set $F(0)$ is not bounded. Hence (1) is not a Filippov DI. □

Obviously, no vector component of an asymptotically stable DI is identical zero. Thus for such DIs $w_i \geq p$ for $i = 1, \ldots, m$.

DI (1) is called *finite-time stable* (FTS), if the origin 0 is a Lyapunov-stable constant solution, and each solution of the DI stabilizes at 0 in FT.

Proposition 2. *Let (1)–(3) define a FTS Filippov homogeneous DI. Then $p > 0$, and $w_i \geq p$ for $i = 1, \ldots, m$.*

Proof. Obviously $\forall i$ $w_i \geq p$. Prove that $p > 0$.

Choose the sphere $S_1 = \{\|s\| = 1\}$ and any $\kappa_0 \in (0, 1)$. Obviously, $d_{\kappa_0} S_1$ lies inside the sphere S_1. Due to its upper-semicontinuity the set function F is bounded on each compact, in particular between S_1 and $d_{\kappa_0} S_1$. Thus there exists a number $T_m > 0$, such that no trajectory starting on S_1 hits $d_{\kappa_0} S_1$ in time less than T_m. Applying the transformation (3) with the parameter κ_0^k to such trajectories implies that for any integer k no trajectory starting on $d_{\kappa_0}^k S_1$ hits $d_{\kappa_0}^{k+1} S_1$ in time less than $\kappa_0^{kp} T_m$.

Any stabilizing trajectory starting on S_1 hits the manifolds $d_{\kappa_0} S_1, d_{\kappa_0}^2 S_1, \ldots$ on its way to 0. Assuming $p \leq 0$, get that the stabilization time is not less than $T_m \sum_k \kappa_0^{kp} = \infty$. □

It is known that the asymptotic stability of homogeneous DIs with negative homogeneity degree, i.e. with $p > 0$, is equivalent to their FT stability (Bhat & Bernstein, 2000; Levant, 2005a; Orlov, 2005).

Let $\Phi(s)$, $s \in \mathbb{R}^m$, be the set of all solutions of (1) defined for $t \geq 0$, with the initial value s at the time $t = 0$.

For any $\xi \in \Phi(s)$, the functional $T_0(\xi) = \inf\{\tau \geq 0 \mid \forall t \geq \tau,\ \xi(t) = 0\}$ is called the *settling-time* of $\xi(t)$. If the set $\{\tau \geq 0 \mid \forall t \geq \tau,\ \xi(t) = 0\}$ is empty then the value $T_0(\xi) = \infty$ is assigned. Note that due to the FT stability of (1), $T_0(\xi)$ is finite, and $\xi(t) = 0$ for all $t \geq T_0(\xi)$.

Introduce the *upper settling-time* function $T^*(s) = \sup\{T_0(\xi) \mid \xi \in \Phi(s)\}$, and the *lower settling-time* function $T_*(s) = \inf\{T_0(\xi) \mid \xi \in \Phi(s)\}$. Obviously, the functions $T^*(s)$ and $T_*(s)$ are homogeneous of the weight p. Indeed, due to the invariance of (1) with respect to the transformation (3) get $T^*(d_\kappa s) = \sup_{\hat{\xi} \in \Phi(d_\kappa s)} T_0(\hat{\xi}) = \sup_{\xi \in \Phi(s)} \kappa^p T_0(\xi) = \kappa^p T^*(s)$. The homogeneity of T_* is similarly proved.

It was erroneously stated (Levant, 2005a) that the maximal convergence time of a FTS homogeneous DI is a continuous function of initial conditions. The following proposition corrects the statement.

Proposition 3. *Let (1) be a homogeneous FTS Filippov DI with the homogeneity dilation (2) and transformation (3). Then the following statements are true.*

1. *The set $\{T_0(\xi) \mid \xi \in \Phi(s)\}$ is compact for any $s \in \mathbb{R}^m$. In particular, $T^*(s) = \max\{T_0(\xi) \mid \xi \in \Phi(s)\}$, $T_*(s) = \min\{T_0(\xi) \mid \xi \in \Phi(s)\}$, i.e., both functions are realized on some solutions of (1).*
2. *The upper settling-time function $T^*(s)$ is an upper semicontinuous function, whereas the lower settling-time function $T_*(s)$ is a lower semicontinuous function.*