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Brief paper

Distributed Nash equilibrium seeking: A gossip-based algorithm☆

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ABSTRACT

This paper presents an asynchronous gossip-based algorithm for finding a Nash equilibrium (NE) of a game in a distributed multi-player network. The algorithm is designed in such a way that players make decisions based on estimates of the other players' actions obtained from local neighbors. Using a set of standard assumptions on the cost functions and communication graph, the paper proves almost sure convergence to a NE for diminishing step sizes. For constant step sizes an error bound on expected distance from a NE is established. The effectiveness of the proposed algorithm is demonstrated via simulation for both diminishing and constant step sizes.

1. Introduction

Finding a NE in a distributed multi-player network game is a problem that has received increasing attention in recent years. Many important real-world applications in wired and wireless networks involve such a setup (Chen & Huang, 2012; Stankovic, Johansson, & Stipanovic, 2012; Yin, Shanbhag, & Mehta, 2011). Peer-to-peer (P2P) and mobile ad-hoc networks are two examples among many. In this problem each player pursues minimization of his cost function selfishly by taking an action in response to other players' actions. This requires full information on all other players' actions in the network. However, this is a stringent requirement in a distributed network. Players have to minimize their cost functions based on limited local information from the neighboring players.

Our goal is to design a locally distributed algorithm to find a NE in a networked continuous kernel game. In such a game, all the players share their information locally and update their actions in order to minimize their cost functions according to the limited information.

Literature review. Our work is related to the literature on Nash games (Alpcan & Başar, 2005; Yin et al., 2011). Distributed algorithms for computing NE have recently drawn significant attention due to a wide range of applications, to name only a few (Alpcan & Başar, 2005; Frihauf, Krstic, & Basar, 2012; Pan & Pavel, 2009; Pavel, 2007). In Kannan and Shanbhag (2010), an iterative regularization algorithm is studied for monotone game. A distributed algorithm for a class of generalized games is proposed in Zhu and Frazzoli (2012) which studies convergence to a NE for a *complete communication graph*. The paper (Gharesifard & Cortes, 2013) considers a distributed algorithm for NE seeking in a *two-network zero-sum game*. A new systematic methodology is presented in Li and Marden (2013) to find distributed algorithms for games with local-agent utility functions (proved to be state-based *potential games*). The algorithms are designed to be dependent on information from only a set of local neighboring agents. The authors in Bramoull, Kranton, and D'Amours (2014) generalize the problem of finding NE (in special games such as those involving strategic innovation, public goods, and social interactions) to the case in which players are considered to be linked if their payoffs are directly affected by the action of the others. A distributed learning algorithm is proposed in Chen and Huang (2012) for finding NE in a *spatial spectrum access game*, albeit for games with finite action spaces. In Gharehshiran, Krishnamurthy, and Yin (2013), regret-based reinforcement learning algorithms have been developed for equilibrium seeking over networks in finite action games. A fictitious play-based approach has been proposed in Swenson, Kar, and Xavier (2012), in which an average empirical distribution is tracked. In Wang et al. (2013) a distributed consensus protocol was proposed for finding a Nash equilibrium of a *congestion game*.

Gossip-based communication has been widely used in asynchronous algorithms due to simplicity and applicability particularly for distributed optimization (Lee & Nedic, 2016; Ram, Nedić, & Veeravalli, 2010). In Koshal, Nedic, and Shanbhag (2012), a gossip-based algorithm has been designed for finding a NE in *aggregative*

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games. Since the algorithm is designed for aggregative games, there is no need to estimate the other players' actions. However, in a general game the aggregate of the players' actions is not enough to update players' actions.

Contributions. Inspired by (Koshal et al., 2012), we propose an asynchronous gossip-based algorithm for a larger class of games. In the proposed algorithm each player maintains an estimate vector of the other players' actions. A communication protocol is designed for sharing estimates and actions between the local players so that they update their estimates and actions. In contrast to Koshal et al. (2012), in which the players take average of the scalar aggregate estimate including their own action, our algorithm excludes the players' actions from their estimates. This exclusion is appropriate in a general game, but it precludes exploiting doubly stochastic properties in the gossiping step. However, we overcome this drawback by using an extra intermediary variable.

In general, convergence properties of distributed algorithms depend on the selection of step sizes (Blatt, Hero, & Gauchman, 2007; Kvaternik & Pavel, 2011; Nedic & Ozdaglar, 2009). Diminishing step sizes typically lead as close as possible to the optimal point (Nedic, 2011), but can have slow convergence, while constant step sizes cause a fluctuation around the optimal point (Zhang, Zheng, & Chiang, 2008). The choice of step size usually demands a trade off between convergence speed and accuracy of convergence. A preliminary version of this work treating only diminishing step sizes has appeared in Salehisadaghiani and Pavel (2014). In this paper we consider both diminishing as well as constant step sizes. Using a set of standard assumptions on the cost functions and communication graph, for diminishing step sizes we prove almost sure (a.s.) convergence toward a NE of the game. For constant step sizes we establish an error bound on the expected distance from the NE.

The paper is organized as follows. In Section 2, the problem statement and assumptions are provided. An asynchronous gossip-based algorithm is proposed in Section 3. In Section 4, convergence of the algorithm with diminishing step sizes is discussed, while in Section 5 constant step sizes are considered. Simulation results are presented in Section 6 and conclusions in Section 7.

1.1. Notation

The $N \times N$ identity matrix and the $N \times 1$ vector of 1's are denoted by I_N and $\mathbf{1}_N$, respectively. We use e_i to denote a unit vector in $\mathbb{R}^N$ whose ith element is 1 and the others are 0. The limit superior of a sequence x_n is defined as $\limsup_{n\to\infty} x_n := \inf_{n\geq 0} \sup_{m\geq n} x_m$.

2. Problem statement

Consider a set of N players in a network specified by a *communication graph* $G_C(V, E_C)$ where $V = \{1, \dots, N\}$ denotes the set of players and $E_C \subset V \times V$ specifies the pairs of players that may communicate. The set of neighbors of player i in G_C, denoted by $N_C(i)$, is the set of vertices which are connected to vertex i by an edge, i.e., $N_C(i) := \{j \in V | (i, j) \in E_C\}$. For $i \in V$, $J_i : \Omega \to \mathbb{R}$ is the cost function of player i where $\Omega = \Omega_i \times \Omega_{-i} \subset \mathbb{R}^N$ is the action set of all players and $\Omega_i \subset \mathbb{R}$ is the action set of player i. The Nash game denoted by $\mathcal{G}(V, \Omega_i, J_i)$ is defined based on the set of players V, the action set Ω_i, $\forall i \in V$ and the cost function J_i, $\forall i \in V$. Let $x = (x_i, x_{-i}) \in \Omega$, with $x_i \in \Omega_i$, denote all players' actions. The cost function J_i depends on all (x_i, x_{-i}). The game is played such that for given $x_{-i} \in \Omega_{-i}$, each player i aims to minimize his own cost function selfishly to find an optimal action,

$$\begin{aligned} \underset{y_i}{\text{minimize}} \quad & J_i(y_i, x_{-i}) \\ \text{subject to} \quad & y_i \in \Omega_i. \end{aligned} \tag{1}$$

Note that the solution set of player i in (1), depends on the actions of the other players x_{-i}. We assume that the cost function J_i and the action set Ω are only available to player i, $i \in V$. Thus players are required to exchange some information to update their actions.

Assumption 1. The communication graph $G_C(V, E_C)$ is connected and undirected.

The connectivity assumption is critical in order to ensure that the information on each player is reached by all other players, infinitely often.

The NE of the game is defined as follows.

Definition 1. Consider an N-player game $\mathcal{G}(V, \Omega_i, J_i)$. A vector $x^* = (x_i^*, x_{-i}^*) \in \Omega$ is called a NE of this game if and only if,

$$J_i(x_i^*, x_{-i}^*) \leq J_i(x_i, x_{-i}^*) \quad \forall x_i \in \Omega_i,\ \forall i \in V. \tag{2}$$

We review next some basic results. A NE can be efficiently computed by solving the associated *Variational Inequality* (*VI*) problem.

Proposition 1 (*Proposition 1.4.2, Facchinei & Pang, 2003*)**.** *Let Ω_i be a closed convex subset of $\mathbb{R}$ for $i \in V$. Let also for $i \in V$, function $J_i(y_i, x_{-i})$ be convex and continuously differentiable in y_i for each fixed x_{-i}. Then a tuple $x^* = (x_i^*, x_{-i}^*)$ is a NE if and only if $x^* \in SOL(\Omega, F)$, where $SOL(\Omega, F)$ is the solution set of $VI(\Omega, F)$, $\Omega = \Omega_i \times \Omega_{-i}$ and $F(x)^T = [\nabla_{x_1}^T J_1(x), \dots, \nabla_{x_i}^T J_i(x), \dots, \nabla_{x_N}^T J_N(x)]$. ($F : \Omega \to \mathbb{R}^N$ is called a pseudo-gradient mapping.)*

Using Proposition 1, one can characterize a NE in terms of a *VI* problem as in the following lemma (Proposition 1.5.8, page 83 in Facchinei & Pang, 2003).

Lemma 1. *x^* is a NE of the game represented by (1) if and only if $x^* = T_\Omega[x^* - \alpha F(x^*)]$ for $\alpha > 0$, where $T_\Omega : \mathbb{R}^N \to \Omega$ is a Euclidean projection.*

In the following, we state a few assumptions including the existence and uniqueness conditions of a NE.

Assumption 2. The set Ω_i is non-empty, compact and convex subset of $\mathbb{R}$ for every $i \in V$. The cost function of player i, $J_i(x_i, x_{-i})$ is a continuously differentiable function in x_i for every $i \in V$. Also $J_i(x_i, x_{-i})$ is jointly continuous in x and convex in x_i for every x_{-i} and $i \in V$.

By Assumption 2, it follows that there exists $C > 0$ such that for all $i \in V$ and for all $x \in \Omega$,

$$\|\nabla_{x_i} J_i(x)\| \leq C. \tag{3}$$

Assumption 3. $F : \Omega \to \mathbb{R}^N$ is strictly monotone on Ω, i.e., $(F(x) - F(y))^T (x - y) > 0\ \forall x, y \in \Omega, x \neq y$.

Assumption 4. $\nabla_{x_i} J_i(x_i, u)$ is Lipschitz continuous in x_i (u), for every fixed $u \in \Omega_{-i}$ ($x_i \in \Omega_i$) and for every $i \in V$, that is, for some positive constant σ_i (L_i), $\|\nabla_{x_i} J_i(x_i, u) - \nabla_{x_i} J_i(y_i, u)\| \leq \sigma_i \|x_i - y_i\|$ $\forall x_i, y_i \in \Omega_i$ ($\|\nabla_{x_i} J_i(x_i, u) - \nabla_{x_i} J_i(x_i, z)\| \leq L_i \|u - z\| \forall u, z \in \Omega_{-i}$).

3. Asynchronous Gossip-based algorithm

We propose an asynchronous gossip-based algorithm to compute a NE of $\mathcal{G}(V, \Omega_i, J_i)$ over $G_C(V, E_C)$ using only partial information. As in Proposition 1, we obtain a NE by solving the associated *VI* problem using a projected gradient-based method.