Automatica 71 (2016) 62–71

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Positive output/state maps and quasi-positive realization of MIMO discrete systems ☆

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ARTICLE INFO

Article history:
Received 16 January 2015
Received in revised form
4 April 2016
Accepted 11 April 2016
Available online 31 May 2016

Keywords:
Realization
Positive systems
Quasi-positive systems
Linear systems
MIMO systems

ABSTRACT

The positive realization of externally positive systems described by their transfer matrix $G(z)$ is a complex problem whose analysis has been completed only recently, for the SISO case, on the basis of algebraic and geometric approaches that underline the many constraints that condition its solution. These constraints concern the minimal order of the obtained models, the minimality of their parameterizations and even their existence. This paper considers the new category of *quasi-positive* state-space models introduced in Guidorzi (2014) that limit the assumptions on the nonnegativeness of the state-space trajectories to the only trajectories that can be actually generated by the system under nonnegative controls. It is shown that all externally positive systems admit a quasi-positive minimally parameterized state-space realization whose existence is not conditioned by restrictions on the signs of the parameters of the polynomials in $G(z)$.

1. Introduction

A large number of real processes are described by positive models i.e. by models that generate nonnegative output trajectories when driven by nonnegative inputs. Well known examples concern, for instance, pharmacokinetics, the diffusion of pollutants in the environment, stocking, chemical and demographic processes as well as many others (Benvenuti & Farina, 2001; Garzia & Lockhart, 1989; Jacquez, 1985; Kajiya, Kodama, & Abe, 1984; Rabinovitz, Wetherill, & Kopple, 1973; Van Schuppen, 1986). This relevance has attracted a wide research interest in the properties of the associated models and in the analysis and control procedures that can be applied to this class of systems.

Modern system and control theory relies almost entirely on the use of state-space models and, consequently, the definition of positivity for these models (Anderson, 1997; Benvenuti, De Santis, & Farina, 2003; Bru & Romero-Vivó, 2009; Farina & Rinaldi, 2000; Kaczorek, 2002a,b; Luenberger, 1979) as well as the analysis of their reachability and controllability, (Benvenuti & Farina, 2006; Coxson & and Shapiro, 1987; Evans & Murthy, 1977; Heemels, van Eijndhoven, & Stoorvogel, 1998; Muratori & Rinaldi, 1989; Murthy, 1986; Rumchev & James, 1989; Valcher, 1996), stability and stabilizability, (Benvenuti & Farina, 2004a; Fornasini & Valcher, 1995; Muratori & Rinaldi, 1991; Rumchev & James, 1995a,b; Son & Hinrichsen, 1996; Valcher & Farina, 2000) and observability, (Van den Hof, 1998) properties play a central role from both theoretical and practical viewpoints and have been deeply investigated.

State-space positive models (also called *internally positive*) are defined as models where any nonnegative initial state generates, with nonnegative input sequences, nonnegative state and output trajectories, i.e. trajectories belonging to the positive orthants of the state and output spaces. Despite this simple and "natural" definition, the positive realization problem proved to be very challenging because of the severe existence conditions that lead also to the impossibility, in some cases, of obtaining realizations and, in some other cases, lead to models whose order is much larger than the order of the associated transfer function (Benvenuti & Farina, 1999). The direct use of these models can lead to computational difficulties in their analysis and synthesis and it is thus important to have the possibility of replacing positive high order models with reduced ones selected on the basis of suitable criteria like those described in Li, Lam, Wang, and Date (2011) and Li, Yu, and and Gao (2015).

A very complete picture of the positive realization problem can be found in the excellent tutorial (Benvenuti & Farina, 2004b) that treats, from both geometrical and algebraic points of view, this topic and the constraints associated with the search for

☆ The material in this paper was partially presented at the 13th European Control Conference, June 24–27, 2014, Strasbourg, France. This paper was recommended for publication in revised form by Associate Editor Denis Arzelier under the direction of Editor Richard Middleton.

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http://dx.doi.org/10.1016/j.automatica.2016.04.024
0005-1098/

minimal state-space realizations. Other interesting contributions can be found in Anderson (1999), Anderson, Deistler, Farina, and Benvenuti (1996), Farina and Benvenuti (1995), Kaczorek (1997), Kaczorek (2011), Kaczorek and Sajewski (2014), Ohta, Maeda, and Kodama (1984), Valcher (2001) and Van den Hof (1997).

A new approach to the realization of externally positive systems has been introduced in Guidorzi (2014) on the basis of the definition of quasi-positive state-space models that limit the positivity properties to reachable states. The introduction of quasi-positive systems allows to prove the existence of positive output/state maps associated with a well defined state-space basis and to use a canonical realization algorithm to obtain a minimal quasi-positive state-space model with the assigned positive transfer function.

It can be observed that relaxing the positivity constraint on non reachable states has no consequences on the capability of the model to describe the behavior of externally positive systems while it assures not only existence conditions but also the minimality of both order and parameterization of the realization.

The analysis performed in Guidorzi (2014) is limited to the SISO case; the purpose of this paper concerns the extension of quasi-positive realization to the MIMO case and also to external descriptions not limited to transfer matrices but including MFD (Matrix Fraction Descriptions) and generic input/output trajectories. While the extension of the definition of quasi-positive systems from the SISO to the MIMO case is straightforward, the proof of the existence of positive maps between output and state trajectories associated with a well defined basis in the state-space is more complex and is based on the properties of suitable classes of canonical MFD and state-space models.

The contents are organized as follows. Section 2 recalls the main differences between positive and quasi-positive models, compares the respective properties and proposes an example concerning a real system that underlines the possible advantages associated with the use of quasi-positive models in realization. Section 3 proves the existence of positive output/state maps for MIMO systems when a suitable basis is selected in the state-space. The previous results are then used in Section 4 for solving the quasi-positive realization problem for MFD models. Section 5 concerns the quasi-positive realization of generic input/output trajectories and Section 6 that of transfer matrices. Short concluding remarks are finally given in Section 7.

2. Positive and quasi-positive models

Systems whose "natural" state is intrinsically constrained in the positive orthant are frequent in econometry, epidemiology, biology, ecology, chemistry, hydraulics, logistics and also in many other fields. Recently some specific properties, like reachability and controllability, that exhibit substantial differences between positive and non positive systems, have been studied with reference to state-space representations and this, in turn, has spurred remarkable interest on the positive realization problem. When the use of positive state-space models to describe externally positive systems is suggested by specific problems, the question to be solved concerns how obtaining models of this kind possibly endowed with additional useful features like minimality.

2.1. Positive state-space models

Positive state-space models could be defined as models where all state and output trajectories generated by nonnegative initial states and input sequences belong to the nonnegative orthant of the state and output spaces. This simple and unambiguous definition implies a long set of properties, usually listed as follows.

Definition 1. Given a $(n \times n)$ nonnegative matrix M, $\sigma(M)$ will denote the spectrum of M, i.e. $\sigma(M) = \mathrm{eig}(M) = \{\lambda_1, \dots, \lambda_n\}$. The *spectral radius* of M, is defined as $\rho(M) = \max_i |\lambda_i|$, $(i = 1, \dots, n)$. Every eigenvalue with maximal modulus is defined as *dominant eigenvalue*, λ_d; thus $\rho(M) = |\lambda_d|$.

In the following the notation $M \geq 0$ will be used for any generic vector or matrix M with nonnegative entries (not to be confused with the common use concerning positive semidefinite matrices). $\mathcal{R}^+$ will denote the positive orthant of R^n.

Properties of positive state-space models

The triple (A, B, C) defining the positive discrete-time state–space model

$$x(k+1) = Ax(k) + Bu(k), \quad (1)$$

$$y(k) = Cx(k) \quad (2)$$

where $x \in \mathcal{X} = \mathcal{R}^n$, $u \in \mathcal{R}^r$, $y \in \mathcal{R}^m$, is endowed with the following properties (Anderson, 1997; Benvenuti et al., 2003; Benvenuti & Farina, 2004b, 2006; Farina & Rinaldi, 2000; Frobenius, 1912; Karpelevich, 1988; Luenberger, 1979; Perron, 1907):

(P1) All states reachable from $x(0) = 0$ for $u(\cdot) \geq 0$ are positive or null.
(P2) For every state $x \in \mathcal{R}^+$, $Ax \in \mathcal{R}^+$.
(P3) The impulse response $H(k) = CA^{(k-1)}B$ is positive or null for every $k > 0$.
(P4) The matrices (A, B, C) have positive or null entries, $A \geq 0$, $B \geq 0$, $C \geq 0$.
(P5) The set of reachable states of system (1)–(2) is the convex pointed reachability cone

$$\mathcal{R}_c = \mathrm{cone}\,\{B,\ AB,\ A^2B,\ \dots\}. \quad (3)$$

(P6) Perron–Frobenius theorem. The dominant eigenvalues of A are all the roots of $\lambda^k - \rho(A)^k = 0$ for some (also more than one) values of $k = 1, 2, \dots, n$. One of the dominant eigenvalues is positive real, i.e. $\rho(A) \in \sigma(A)$ and, for every dominant eigenvalue, λ_d, $\deg \rho(A) > \deg \lambda_d$.
(P7) Karpelevich theorem. The regions of the complex plane that contain the eigenvalues of A are symmetric with respect to the real axis, are included in the disc $|z| \leq 1$, and intersect the circle $|z| = 1$ in the points $e^{(2\pi ia/b)}$ where a and b run over the relatively prime integers satisfying the condition $0 \leq a \leq b \leq n$. The boundary of these regions consists of these points and of curvilinear arcs connecting them in circular order.

2.2. Quasi-Positive state-space models

Quasi-positive state-space models, introduced in Guidorzi (2014), can be described as models where all state and output trajectories generated by *reachable* nonnegative initial states with nonnegative input sequences are nonnegative. The difference with the definition of positive systems is minimal (the word *reachable*), the consequences are not. The properties of quasi-positive models can be listed as follows.

(QP1) All states reachable from $x(0) = 0$ for $u(\cdot) \geq 0$ are positive or null.
(QP2) For every *reachable* state $x \in \mathcal{R}^+$, $Ax \in \mathcal{R}^+$.
(QP3) The impulse response $H(k) = CA^{(k-1)}B$ is positive or null for every $k > 0$.
(QP4) The matrices B and C have positive or null entries, $B \geq 0$, $C \geq 0$.
(QP5) The set of reachable states is the convex pointed reachability cone $\mathcal{R}_c$ (3).