Automatica 71 (2016) 72–77

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Razumikhin-type theorems for hybrid system with memory[☆]

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ARTICLE INFO

Article history:
Received 22 February 2014
Received in revised form
29 November 2015
Accepted 17 April 2016
Available online 31 May 2016

Keywords:
Hybrid systems with memory
Delay systems
Razumikhin-type conditions
Pre-asymptotically stable

ABSTRACT

In this paper, we address the stability problem of hybrid systems with memory, which are newly developed to model hybrid dynamic systems affected by delays. For such systems, stability criteria are developed based on Razumikhin-type conditions, which are usually applied in the delay systems. We prove that the hybrid system with memory is uniformly globally pre-asymptotically stable by small gain theorem. Besides, a relaxed Razumikhin-type stability criterion is also proposed which does not require strict decrease of the Lyapunov–Razumikhin function during jumps. Several examples are given to illustrate the effectiveness and generality of the proposed results.

1. Introduction

A hybrid system is a dynamic system that exhibits characteristics of both continuous-time and discrete-time dynamical systems (Goebel, Hespanha, Teel, Cai, & Sanfelice, 2004; Goebel, Sanfelice, & Teel, 2012; Goebel & Teel, 2006; Nesic & Teel, 2004). Stability theorems have been well developed for hybrid systems by Lyapunov function (Goebel et al., 2012; Goebel, Sanfelice, & Teel, 2009) based on the framework introduced in Goebel and Teel (2006). Also, it is well known that time delays are often inevitable in many physical systems and may degrade the system performance and even lead to the instability of the systems (Cloosterman, van de Wouw, Heemels, & Nijmeijer, 2008; Gu, Kharitonov, & Chen, 2003; Hale & Lunel, 1993; Heemels, Teel, van de Wouw, & Nesic, 2010; van de Wouw, Nešić, & Heemels, 2012). Therefore, stability analysis for control systems with delays is a crucial problem and has attracted a lot of attention (see, for example, Fridman and Shaked (2003), Gu et al. (2003), Hale and Lunel (1993), He, Wang, Lin, and Wu (2007), Niculescu (2001), Pepe and Jiang (2006), Sun, Zhao, and Hill (2006), Sun and Wang (2012), Teel (1998), and references cited therein).

The newly developed hybrid system with memory refers to dynamic system exhibiting both hybrid and delay phenomena (Liu & Teel, 2012, 2014). The pioneering papers Liu and Teel (2012, 2014) extend the generalized solution for hybrid systems to hybrid systems with memory. The proposed framework in Liu and Teel (2012, 2014) possesses nice structure properties and makes it possible to investigate more profound and general stability theory involved with properties of solutions.

Razumikhin approach is one of the most important theoretical tools to analyze the stability of ordinary nonlinear functional differential equations and has advantage over Lyapunov–Krasovskii functional approach due to its easy construction (Hale & Lunel, 1993). Several extensions and variations of such approach to discrete delay systems are carried out in Elaydi and Zhang (1994) and Liu and Marquez (2007) where Elaydi and Zhang (1994) provides noncausal conditions, while the conditions in Liu and Marquez (2007) are causal. Different from the continuous delay systems and discrete delay systems, the impulsive delay systems (Ballinger & Liu, 1999, 2000) are a class of important hybrid delay systems and can only admit one jump at each jump instant. Stability analysis for such systems is more challenging since the stability conditions heavily rely on the continuous dynamic and discrete dynamic. By admitting various kinds of constraints on Lyapunov–Razumikhin function, several stability conditions are proposed to deal with the stability problems of impulsive delay systems in a series of papers (Liu, 2004; Liu & Ballinger, 2001; Shen & Yan, 1998; Stamova & Stamov, 2001; Wang & Liu, 2005). However, all the aforementioned work is restricted to be a special case of hybrid systems with memory in Liu and Teel (2012, 2014), where not only multiple jumps

☆ This work was supported by the National Natural Science Foundation of China under Grant No. 61325014. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Maurice Heemels under the direction of Editor Andrew R. Teel.

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http://dx.doi.org/10.1016/j.automatica.2016.04.038

are permitted at some instant but also functional differential inclusions and functional difference inclusions are included. With these new introduced problems and difficulties, the existing tools and techniques cannot be applied to the new systems directly except the stability criterion for hybrid systems with memory in Liu and Teel (2014, 2016) by Lyapunov–Razumikhin function.

We propose several new Razumikhin-type stability conditions for hybrid systems with memory in the framework of Liu and Teel (2014). In order to overcome the problems resulted from the hybrid characteristic and delay characteristic, a comparison lemma of hybrid version is developed and the key technique adopted in the proof is the hybrid small gain theorem inspired by Teel (1998) that deals with the continuous delay systems. The stability conditions unify the Razumikhin-type stability criteria for continuous delay systems (Hale & Lunel, 1993) and discrete delay systems (Elaydi & Zhang, 1994). Compared with the Razumikhin-type stability criterion proposed in Liu and Teel (2016), we adopt a direct interpretation of Razumikhin theorem for discrete systems (Elaydi & Zhang, 1994) and a different technique based on small gain theorem is offered. Besides, a relaxed stability criterion is also proposed, which does not require the strict decrease of the Lyapunov–Razumikhin function during jumps. Several examples are also given to illustrate the generality and effectiveness of the obtained results.

Throughout this paper, the following notation is used.

$\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{Z}$ denotes the integer set. Define $\mathbb{R}_{\geq 0} = [0, +\infty)$, $\mathbb{R}_{\leq 0} = (-\infty, 0]$, $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}_{\leq 0} = \{0, -1, -2, \ldots\}$. For any $x \in \mathbb{R}^n$ and a closed subset $\mathcal{W}$ of $\mathbb{R}^n$, $|x|_{\mathcal{W}} \triangleq \inf_{y \in \mathcal{W}} |x - y|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be $\mathcal{K}_\infty$ if it is zero at zero, continuous, strictly increasing, and unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ function, if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r \to 0^+} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s \to \infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. For a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ and a direction vector $\upsilon \in \mathbb{R}^n$, define $V^\circ(x, \upsilon) = \limsup_{h \downarrow 0, y \to x} \frac{V(y+h\upsilon)-V(y)}{h}$.

2. Preliminaries

In this section, we will review some basic knowledge related to hybrid systems with memory from the literature (Liu & Teel, 2012, 2014).

Definition 1 (*Liu and Teel (2014)*)**.** A subset $E \subseteq \mathbb{R} \times \mathbb{Z}$ is called a compact hybrid time domain with memory if $E = E_{\geq 0} \bigcup E_{\leq 0}$, where

$$E_{\geq 0} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j), \qquad E_{\leq 0} = \bigcup_{k=1}^{K} ([s_k, s_{k-1}], -k+1) \tag{1}$$

for some finite sequence of times $s_K \leq \cdots \leq s_1 \leq s_0 = 0 = t_0 \leq t_1 \leq \cdots \leq t_J$. The set E is called a hybrid time domain with memory if, for all $(T, J) \in E_{\geq 0}$ and all $(S, K) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, $(E_{\geq 0} \bigcap ([0, T] \times \{0, 1, \ldots, J\})) \bigcup (E_{\leq 0} \bigcap ([-S, 0] \times \{-K, -K+1, \ldots, 0\}))$ is a compact hybrid time domain with memory.

A function $x : E \to \mathbb{R}^n$ is said to be a hybrid arc with memory if E is a hybrid time domain with memory and for each $j \in \mathbb{Z}$, the function $x(t, j)$ is locally absolutely continuous on $I_j = \{t : (t, j) \in E\}$. Given a hybrid arc with memory x, denote the domain of x by $\mathrm{dom}x$ and $\mathrm{dom}x \cap (\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0})$ by $\mathrm{dom}_{\geq 0}x$. A hybrid arc with memory x is said to be a hybrid memory arc if $\mathrm{dom}x \subseteq \mathbb{R}_{\leq 0} \times \mathbb{Z}_{\leq 0}$.

The notation $\mathcal{M}$ denotes the space of hybrid memory arcs. The positive length of a hybrid time domain with memory E is defined as $\mathrm{length}_{\geq 0}(E) = \sup_{t \geq 0} E + \sup_{j \geq 0} E$, where $\sup_{t \geq 0} E = \sup\{t : \exists j \in \mathbb{Z}_{\geq 0}$ *such that* $(t, j) \in \mathrm{dom}_{\geq 0}x\}$ and $\sup_{j \geq 0} E = \sup\{j : \exists t$ *such that* $(t, j) \in \mathrm{dom}_{\geq 0}x\}$. $\mathrm{length}_{\leq 0}(E)$ can be defined similarly.

Given $\Delta > 0$, the notation $\mathcal{M}^\Delta$ denotes the collection of the hybrid memory arc $\varphi \in \mathcal{M}$ satisfying $-\Delta - 1 \leq \mathrm{length}_{\leq 0}(\mathrm{dom}\varphi) \leq -\Delta$.

Given a hybrid arc with memory x, the operator $\mathcal{A}^\Delta_{[\cdot,\cdot]}x : \mathrm{dom}_{\geq 0}x \to \mathcal{M}^\Delta$ is defined by $\mathcal{A}^\Delta_{[t,j]}x(s, k) = x(t+s, j+k)$ for all $(s, k) \in \mathrm{dom}\mathcal{A}^\Delta_{[t,j]}x$, where $(t, j) \in \mathrm{dom}_{\geq 0}x$, $\mathrm{dom}\mathcal{A}^\Delta_{[t,j]}x = \{(s, k) \in \mathbb{R}_{\leq 0} \times \mathbb{Z}_{\leq 0} : (t+s, j+k) \in \mathrm{dom}x, s+k \geq -\Delta_{\inf}\}$, and $\Delta_{\inf} = \inf\{\delta \geq \Delta : \exists (t+s, j+k) \in \mathrm{dom}x \; s.t. \; s+k = -\delta\}$. For a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ and a hybrid arc with memory x satisfying $\mathrm{length}_{\leq 0}(\mathrm{dom}\mathcal{A}_{[0,0]}x) \leq -\Delta - 1$ for $\Delta > 0$, denote $\bar{V}_{[t,j]}(x) = \sup_{-\Delta-1 \leq s+k \leq 0} V(x(t+s, j+k))$ for each $(t, j) \in \mathrm{dom}_{\geq 0}x$.

$\mathcal{H}^\Delta_{\mathcal{M}} = (\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$ is said to be a hybrid system with memory with size $\Delta > 0$ where $\mathcal{C}$, $\mathcal{F}$, $\mathcal{D}$ and $\mathcal{G}$ are data of $\mathcal{H}^\Delta_{\mathcal{M}}$. Specifically, $\mathcal{C} \subset \mathcal{M}^\Delta$ is the flow set, $\mathcal{D} \subset \mathcal{M}^\Delta$ is the jump set, the set-valued functional $\mathcal{F} : \mathcal{M}^\Delta \rightrightarrows \mathbb{R}^n$ is the flow map, and the set-valued functional $\mathcal{G} : \mathcal{M}^\Delta \rightrightarrows \mathbb{R}^n$ is the jump map. A hybrid arc with memory x is a solution to $\mathcal{H}^\Delta_{\mathcal{M}}$ if $\mathcal{A}^\Delta_{[0,0]}x \in \mathcal{C} \cup \mathcal{D}$ and, (S1) for all $j \in \mathbb{Z}_{\geq 0}$ and almost all t such that $(t, j) \in \mathrm{dom}x$, $\mathcal{A}^\Delta_{[t,j]}x \in \mathcal{C}$ and $\dot{x}(t, j) \in \mathcal{F}(\mathcal{A}^\Delta_{[t,j]}x)$, (S2) for all $(t, j) \in \mathrm{dom}_{\geq 0}x$ such that $(t, j+1) \in \mathrm{dom}_{\geq 0}x$, $\mathcal{A}^\Delta_{[t,j]}x \in \mathcal{D}$ and $x(t, j+1) \in \mathcal{G}(\mathcal{A}^\Delta_{[t,j]}x)$.

On the existence of the solutions and the well-posedness of the system $\mathcal{H}^\Delta_{\mathcal{M}}$, details can be found in Liu and Teel (2014, 2016, submitted for publication). The set of all maximal solutions to $\mathcal{H}^\Delta_{\mathcal{M}}$ is denoted by $\mathcal{S}_{\mathcal{H}^\Delta_{\mathcal{M}}}$.

3. Stability of hybrid systems with memory

In this section, we will give some stability concepts for hybrid systems with memory and present our main results. An equivalent definition of uniform global pre-asymptotic stability for $\mathcal{H}^\Delta_{\mathcal{M}}$ based on class-$\mathcal{KL}$ function is given in Liu and Teel (2014).

Definition 2. Let $\mathcal{H}^\Delta_{\mathcal{M}}$ be a hybrid system with memory in $\mathcal{M}^\Delta$. A closed set $\mathcal{W} \subseteq \mathbb{R}^n$ is said to be

- uniformly globally stable for $\mathcal{H}^\Delta_{\mathcal{M}}$ if there exists a function $\alpha \in \mathcal{K}_\infty$ such that any solution x to $\mathcal{H}^\Delta_{\mathcal{M}}$ satisfies $|x(t, j)|_{\mathcal{W}} \leq \alpha(|\mathcal{A}_{[0,0]}x|^\Delta_{\mathcal{W}})$ for all $(t, j) \in \mathrm{dom}_{\geq 0}x$;
- uniformly globally pre-attractive for $\mathcal{H}^\Delta_{\mathcal{M}}$ if for any $r > 0$ and $\epsilon > 0$, there exists $T(r, \epsilon) > 0$ such that for any solution x to $\mathcal{H}^\Delta_{\mathcal{M}}$ with $|\mathcal{A}_{[0,0]}x|^\Delta_{\mathcal{W}} \leq r$, $(t, j) \in \mathrm{dom}x$ and $t+j \geq T(r, \epsilon)$ imply $|x(t, j)|_{\mathcal{W}} \leq \epsilon$;
- uniformly globally pre-asymptotically stable (UGpAS) for $\mathcal{H}^\Delta_{\mathcal{M}}$ if it is both uniformly globally stable and uniformly globally pre-attractive,

where $|\mathcal{A}_{[0,0]}x|^\Delta_{\mathcal{W}} = \sup_{-\Delta-1 \leq s+k \leq 0} |\mathcal{A}_{[0,0]}x(s, k)|_{\mathcal{W}}$.

The following definition of $\mathcal{PD}$ function comes from Definition 3.17 in Chapter 3 of Goebel et al. (2012).

Definition 3. A function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is positive definite, also written $\rho \in \mathcal{PD}$, if $\rho(s) > 0$ for all $s > 0$ and $\rho(0) = 0$.

The following stability criterion unifies the Razumikhin-type stability conditions for continuous delay systems and discrete delay systems in the framework of hybrid systems with memory.

Denote $\mathcal{G}^+(\mathcal{D}) := \{\varphi^+_g : g \in \mathcal{G}(\varphi), \varphi \in \mathcal{D}\}$ where φ^+_g is a hybrid memory arc satisfying $\varphi^+_g(0, 0) = g$ and $\varphi_g(s, k-1) = \varphi(s, k)$ for all $(s, k) \in \mathrm{dom}\varphi$.

Theorem 1. *Let $\mathcal{H}^\Delta_{\mathcal{M}} = (\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G})$ be a hybrid system with memory with size $\Delta > 0$ and $\mathcal{W} \subset \mathbb{R}^n$ be closed. If there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, a continuous function $\rho \in \mathcal{PD}$, a continuous nondecreasing function $h(s)$ with $0 < h(s) < s$ for $s > 0$, and a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that*