Automatica 71 (2016) 98–105

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Continuous-time probabilistic ultimate bounds and invariant sets: Computation and assignment ☆

Ernesto Kofman [a], José A. De Doná [b], Maria M. Seron [b], Noelia Pizzi [a]

[a] CIFASIS - CONICET, FCEIA - UNR, Argentina

[b] School of Electrical Engineering and Computer Science and CDSC, The University of Newcastle, Australia

ARTICLE INFO

Article history:
Received 13 January 2015
Received in revised form
10 December 2015
Accepted 17 April 2016
Available online 31 May 2016

Keywords:
Invariant sets
Ultimate bounds
Stochastic differential equations
Linear systems
Probabilistic methods

ABSTRACT

The concepts of ultimate bounds and invariant sets play a key role in several control theory problems, as they replace the notion of asymptotic stability in the presence of unknown disturbances. However, when the disturbances are unbounded, as in the case of Gaussian white noise, no ultimate bounds nor invariant sets can in general be found. To overcome this limitation we introduced, in previous work, the notions of probabilistic ultimate bound (PUB) and probabilistic invariant set (PIS) for discrete-time systems. This article extends the notions of PUB and PIS to continuous-time systems, studying their main properties and providing tools for their calculation. In addition, the use of these concepts in robust control design by covariance assignment is presented.

1. Introduction

Dynamical systems under the influence of non-vanishing unknown disturbances cannot achieve asymptotic stability in general. However, under certain conditions, the ultimate boundedness of the system trajectories can be guaranteed and invariant sets can be found. Consequently, the notions of ultimate bounds (UB) and invariant sets (IS) play a key role in control systems theory and design.

A necessary condition to ensure the existence of ultimate bounds and invariant sets is that the disturbances must be bounded. However, in systems theory, disturbances are often represented by unbounded signals such as Gaussian white noise, in which case ultimate bounds and invariant sets cannot be obtained in a classical sense. To overcome this problem, the authors have introduced in Kofman, De Doná, and Seron (2011, 2012) the notions of *probabilistic ultimate bound* (PUB) and *probabilistic invariant set* (PIS), as sets where the trajectories converge to and stay in with a given probability.

Classic UB and IS are an important tool in modern treatments of model predictive control (see, e.g., Mayne, Rawlings, Rao, & Scokaert, 2000; Rawlings & Mayne, 2009), fault diagnosis and fault tolerant control (see, e.g., Olaru, De Doná, Seron, & Stoican, 2010; Seron, Zhuo, De Doná, & Martínez, 2008) and several other applications of set invariance in control problems (see Blanchini, 1999 and the references therein). With the usage of the PUB and PIS notions, many of these applications can be extended to consider also the presence of unbounded disturbances. In fact, some recent works on model predictive control use concepts that are related to probabilistic invariant sets (see, e.g., Cannon, Kouvaritakis, & Wu, 2009; González et al., 2014; Hashimoto, 2013).

Although the concepts in Kofman et al. (2011, 2012) are limited to the discrete-time domain, ultimate boundedness and invariance are also important concepts in continuous-time systems, and they experience the same limitations regarding unbounded disturbances.

Motivated by these facts, this work firstly extends the notions, properties and tools for PUB and PIS developed in Kofman et al. (2011, 2012) to the continuous-time domain. While in the case of PUB the extension is almost straightforward, the concept of probabilistic invariance in continuous time needs to be redefined because of the limitations imposed by the infinite-bandwidth nature of continuous-time white noise disturbances (see, e.g., the insightful discussions in Åström, 1970).

Finally, the problem of designing a feedback controller so that the closed-loop system under white noise disturbances has a

☆ The material in this paper was partially presented at the 19th IFAC World Congress, August 24–29, 2014, Cape Town, South Africa. This paper was recommended for publication in revised form by Associate Editor Akira Kojima under the direction of Editor Ian R. Petersen.

E-mail addresses: kofman@cifasis-conicet.gov.ar (E. Kofman), Jose.Dedona@newcastle.edu.au (J.A. De Doná), Maria.Seron@newcastle.edu.au (M.M. Seron), pizzi@cifasis-conicet.gov.ar (N. Pizzi).

http://dx.doi.org/10.1016/j.automatica.2016.04.041
0005-1098/

desired PUB is addressed. Preliminary results covering only single input systems in controller canonical form were presented by the authors in the conference paper (Kofman, De Doná, Seron, & Pizzi, 2014). The current journal version completes the contribution by presenting new results that generalise the techniques to multiple input systems given in general form.

The paper is organised as follows: Section 2 introduces the concepts of continuous time PUB and PIS and establishes their basic properties. Then, Section 3 presents closed-form formulas for the calculation of PUB and PIS, respectively. Section 4 develops the technique for control design and Section 5 illustrates the results with a numerical example.

2. Background and definitions

We consider a continuous-time LTI system given by the following stochastic differential equation

$$\mathrm{d}x(t) = Ax(t)\mathrm{d}t + \mathrm{d}w(t) \tag{1}$$

with $x(t), w(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ being a Hurwitz matrix.

Assumption 1. The disturbance $w(t)$ is a stochastic process whose increments are stationary and uncorrelated with zero mean values (i.e., a Lévy process, that in the case of a normal distribution becomes a Wiener process). We assume also that $w(t)$ has incremental covariance $\Sigma_w \mathrm{d}t \triangleq \mathrm{cov}[\mathrm{d}w(t)] = \mathrm{E}[\mathrm{d}w(t)\mathrm{d}w^T(t)]$ with Σ_w being a finite covariance matrix.

2.1. Expected Value and Covariance of $x(t)$

The characterisation of probabilistic ultimate bounds and invariant sets is based on the stochastic properties of the solution $x(t)$ of Eq. (1). Given a time t, the covariance of the solution is defined as

$$\Sigma_x(t) \triangleq \mathrm{cov}[x(t)] = \mathrm{E}[(x(t) - E[x(t)])(x(t) - E[x(t)])^T]. \tag{2}$$

Both, Σ_w and $\Sigma_x(t)$ are symmetric positive semidefinite matrices. The expected value $\mu_x(t) = \mathrm{E}[x(t)]$ can be computed (see e.g. Åström, 1970, Theorem 6.1, page 66) as the solution of $\dot{\mu}_x(t) = A\mu_x(t)$. We assume that the initial state $x(t_0)$ is known, then $\mu_x(t_0) = x(t_0)$ and the previous equation has the solution

$$\mu_x(t) = e^{A(t-t_0)}x(t_0). \tag{3}$$

The covariance matrix $\Sigma_x(t)$ verifies (see e.g. Åström, 1970, Theorem 6.1, page 66) the following differential equation:

$$\dot{\Sigma}_x(t) = A\Sigma_x(t) + \Sigma_x(t)A^T + \Sigma_w \tag{4}$$

with $\Sigma_x(t_0) = 0$ (since $x(t_0)$ is known). Since A is a Hurwitz matrix, the latter expression converges as $t \to \infty$. Then, defining $\Sigma_x \triangleq \lim_{t\to\infty} \Sigma_x(t)$ we have from Eq. (4) that Σ_x can be obtained from the Lyapunov equation

$$A\Sigma_x + \Sigma_x A^T = -\Sigma_w. \tag{5}$$

2.2. Definition of PUB and γ-PIS

We next define the two notions that concern this article.

Definition 2 (*Probabilistic Ultimate Bounds*)**.** Let $0 < p \le 1$ and let $S \subset \mathbb{R}^n$. We say that S is a PUB with probability p for system (1) if for every initial state $x(t_0) = x_0 \in \mathbb{R}^n$ there exists $T = T(x_0) \in \mathbb{R}$ such that the probability[1] $\Pr[x(t) \in S] \ge p$ for each $t \ge t_0 + T$.

For the definition of PIS, we first introduce the product of a scalar $\gamma \ge 0$ and a set S as $\gamma S \triangleq \{\gamma x : x \in S\}$. Notice that when $0 \le \gamma \le 1$, and provided that S is a *star-shaped set with respect to the origin*,[2] it follows that $\gamma S \subseteq S$.

Definition 3 (*γ-Probabilistic Invariant Sets*)**.** Let $0 < p \le 1$, $0 < \gamma \le 1$ and let $S \subset \mathbb{R}^n$ be a star-shaped set with respect to the origin. We say that S is a γ-PIS with probability p for system (1) if for any $x(t_0) \in \gamma S$ the probability $\Pr[x(t) \in S] \ge p$ for each $t > t_0$.

Remark 4. The definitions of PUB for discrete and continuous time systems are almost identical. However, PIS for discrete-time systems were defined to ensure that any trajectory starting in the set remains in the set with a given probability. By choosing a sufficiently large set, the contractivity of the system's dynamics at the boundary of the set dominates the noise and the probability of the trajectory leaving the set at the next step can be made arbitrarily small. In continuous time, however, this is not possible. Irrespective of the contractivity, when a trajectory starts at time t_0 at the boundary of the set, taking t sufficiently close to t_0 the dynamics is always dominated by the white noise due to its infinite-bandwidth nature. Thus, for $t \to t_0^+$ the probability that $x(t)$ leaves the set S only depends on the noise and becomes independent of the size of S. In order to overcome this fundamental difficulty, the initial states of a PIS are restricted in Definition 3 to a subset γS, with γ less than one.

The previous remark can be simply illustrated by the solution of the scalar case of Eq. (1) with $w(t)$ a Wiener process and $A = -\lambda$, in which case $x(t) = e^{-\lambda(t-t_0)}x(t_0) + \int_{t_0}^t e^{-\lambda(t-\tau)}\mathrm{d}w(\tau)$. Then, it can be shown that $\lim_{t\to t_0^+} \Pr[|x(t)| > |x(t_0)|] = \lim_{t\to t_0^+} \Pr\left[\int_{t_0}^t \mathrm{d}w(\tau) > 0\right] = 0.5$ independently of $x(t_0)$ and λ (since $\int_{t_0}^t \mathrm{d}w(\tau)$ is a zero-mean Gaussian process). That is, no matter *how contractive* the term $e^{-\lambda t}$ is, nor *how big* the initial condition $|x(t_0)|$ is, the probability of confinement in $|x(t)| \le |x(t_0)|$ *is dominated* by the noise.

2.3. Some properties of PUB and γ-PIS

Here we present some basic properties of PUB and γ-PIS that are analogous to those of deterministic ultimate bounds and invariant sets. Although these properties are not used to derive the main results of the paper, they corroborate that the definitions of PUB and γ-PIS provided above are consistent with their deterministic counterparts.

The basic properties of continuous-time PUB are identical to the discrete-time ones, i.e., Lemma 3 and Corollaries 7 and 10 in Kofman et al. (2012) are also valid for continuous-time PUB. These properties establish that a PUB with probability p for (1) is also a PUB with probability $\tilde{p} \ge 0$ for any $\tilde{p} < p$ and that the union and intersection of PUB sets define PUB sets.

In the case of the unions and intersections of γ-PIS, the presence of the parameter γ introduces some changes to their discrete time counterparts. Lemma 4 in Kofman et al. (2012) is still valid (a γ-PIS with probability p is also PUB with the same probability) but the union and intersection of γ-PIS are now ruled by the following proposition.

Proposition 5 (*Intersection and Union of γ-PIS*)**.** *Let $\{S_i\}_{i=1}^r$ be a collection of γ_i-PIS for system (1) with probabilities p_i, $i = 1, \ldots, r$, respectively, then*

[1] In this work, the expression $\Pr[x(t) \in S \subset \mathbb{R}^n]$ denotes the probability that the solution $x(t)$, at time t, is in the set $S \subset \mathbb{R}^n$. Thus, $\Pr[\cdot]$ is the probability measure on Euclidean space induced by the stochastic process $\{w(\tau)|t_0 \le \tau \le t\}$ via the solution, at time t, of the stochastic differential equation (1) with known initial condition $x(t_0)$ at time t_0.

[2] A set $S \subset R^n$ is star shaped, or a star domain, with respect to the origin if $x \in S \Rightarrow \gamma x \in S$ for all $0 \le \gamma \le 1$.