Brief paper

On synchronization of networks of Wilson–Cowan oscillators with diffusive coupling☆

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ARTICLE INFO

ABSTRACT

We investigate the problem of synchronization in a network of homogeneous Wilson–Cowan oscillators with diffusive coupling. Such networks can be used to model the behavior of populations of neurons in cortical tissue, referred to as neural mass models. A new approach is proposed to address conditions for local synchronization for this type of neural mass models. By analyzing the linearized model around a limit cycle, we study synchronization within a network with direct coupling. We use both analytical and numerical approaches to link the presence or absence of synchronized behavior to the location of eigenvalues of the Laplacian matrix. For the analytical part, we apply two-time scale averaging and the Chetaev theorem, while, for the remaining part, we use a recently proposed numerical approach. Sufficient conditions are established to highlight the effect of network topology on synchronous behavior when the interconnection is undirected. These conditions are utilized to address points that have been previously reported in the literature through simulations: synchronization might persist or vanish in the presence of perturbation in the interconnection gains. Simulation results confirm and illustrate our results.

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1. Introduction

Synchronization is a ubiquitous phenomenon observed in diverse networks of interconnected subsystems that arise in neuroscience, physics, biology, social networks, and many more. Synchronization occurs when the states or outputs of subsystems converge to the same behavior, and can be considered as the asymptotic stability of error vectors between the state (or output) vectors of two or more subsystems. Approaches for the study of synchronization can be categorized into two groups: global and local. Lipschitz (DeLellis, di Bernardo, & Russo, 2011), passivity (Arcak, 2010), dissipativity (Stan & Sepulchre, 2007), and semi-passivity (Steur, Tyukin, & Nijmeijer, 2009) properties have been employed to study global synchronization. In these approaches, the subsystems in the network are required to satisfy a specific dissipation or passivity property. In some applications, it may be challenging to demonstrate that these properties are satisfied, and so local approaches are a useful alternative.

In local approaches, a linearization technique is utilized that indicates that synchronization in a network of oscillators can be analyzed via the well-known master stability equation (MSE) in which the eigenvalues of the Laplacian matrix play a crucial role (Pecora & Carroll, 1998). In order to study the influence of interconnection gains on synchronization, the stability of the MSE has previously been evaluated using numerical approaches, which are computationally intensive. However, a possible way to reduce computational effort would be to combine analytical methods with numerical tools. In Yu, Chen, and Cao (2011), synchronization of a network of oscillators with nonlinear dynamics was investigated analytically. Recently, Shafi, Arcak, Jovanović, and Packard (2013) proposed a framework to study synchronization in a network of oscillators by combining both analytical and numerical methods, allowing one to study the effects of interconnection gains on synchronization.

Synchronization of neural networks is thought to play a key role in information integration and processing. Synchronization of distributed brain regions has been speculated to play an important

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role in cognition (Rodriguez et al., 1999). Therefore, understanding the mechanisms that underpin synchrony in the brain is important. The Wilson–Cowan model (Wilson & Cowan, 1972) is of great interest since it is parsimonious, as it describes the activity of both excitatory and inhibitory populations of neurons and reproduces self-sustained oscillations observed in electroencephalography (EEG) signals. In particular, local synchronization of the Wilson–Cowan model has been investigated in the literature using the center manifold theorem (Hoppensteadt & Izhikevich, 1997) and the notion of phase response curves (Daffertshofer & van Wijk, 2011). These approaches only deal with weak couplings. However, synchronization can be observed in Wilson–Cowan networks with intermediate or strong coupling.

It is known that two factors have a significant impact on presence or absence of synchronization in the complex network: dynamical models of network nodes and network topology. In particular, investigating the effect of the latter has attracted much research and it is still an ongoing problem (Belykh, Belykh, & Hasler, 2006; Porfiri & Di Bernardo, 2008). More recently, this point has been explored in neuroscience using a computational model of the brain (Terry, Benjamin, & Richardson, 2012). It has been observed that, in the network of oscillators, removing or adding interconnections between nodes can lead to the disappearance or persistence of synchronous activity in the system. However, all these observations have been obtained only by simulations.

In this manuscript, we demonstrate that the framework of Shafi et al. (2013) can be adapted to study local synchronization in a network of Wilson–Cowan oscillators with arbitrary coupling strengths. As far as we are aware, this is a new result. Our contribution is fourfold. First, the Wilson–Cowan model does not fit the general model considered in Pecora and Carroll (1998), Shafi et al. (2013) and Yu et al. (2011). As a consequence, the analysis is different. Second, the Wilson–Cowan networks do not synchronize for all coupling gains. Therefore, we had to use an instability result for the linearized model based on the Chetaev theorem to develop a novel proof. This is different from the results in Shafi et al. (2013), where local synchronization was shown for both weak and strong coupling. Furthermore, our results are also different from Yu et al. (2011), where the authors presented a sufficient condition for synchronization that is conservative for our network. Third, we considered the directed coupling between oscillators in the network and our results are general. Fourth, we present sufficient conditions that relate the role of perturbations in the network topology, thereby explaining robustness and absence of synchronization.

The paper is organized as follow. In Section 2, we briefly introduce the Wilson–Cowan model of a single population as well as the network of such models with non-identical nodes. In Section 3, we formulate the problem for a more general network with identical nodes. In Section 4, the synchronization conditions are established for the network. Robustness of synchronization is analyzed in Section 5. Simulation results and conclusions are presented in Sections 6 and 7, respectively.

Notation. Throughout this paper, I_n denotes the identity matrix in $\mathbb{R}^{n \times n}$. The Kronecker product is denoted by $\otimes$. For a complex variable, vector or matrix, $\Re(\cdot)$ and $\Im(\cdot)$ stand for the real and imaginary parts. For a matrix $A \in \mathbb{R}^{N \times N}$, $\{\lambda_i(A)\}_{i=1}^N$ stands for ordered eigenvalues of matrix A such that $\lambda_{min} = \lambda_1$ and $\Re(\lambda_1) \leq \Re(\lambda_2) \leq \cdots \leq \Re(\lambda_N)$. The operator diag$(\cdot)$ constructs a block diagonal matrix from its arguments. $[A]_i$ denotes the ith row of matrix $A \in \mathbb{R}^{N \times M}$.

2. Wilson–Cowan model

Neural mass models describe the relationships between neural populations. Lumped parameter neural mass models are constructed by interconnecting neural populations that generate some realistic EEG patterns like alpha or beta waves. In this class of model, the dynamics of each neural population can be described by a linear first-order system coupled with a sigmoid nonlinearity that converts the average membrane potential of a neural population into an average pulse density of action potentials. This model is given by

$$\dot{x}_s = -\alpha x_s + f(\rho + I), \tag{1}$$

where $x_s \in \mathbb{R}$ describes the average membrane potential of a single population that can be either excitatory x_E or inhibitory x_I. The parameter α is the population time constant and ρ denotes the sensory input or input from other neurons. The inputs from neighboring or distant populations are represented by I. $f_i : \mathbb{R} \to \mathbb{R}$ is a sigmoid function given by

$$f_i(\theta_j) = \frac{1}{1 + \exp(-r_i \theta_j)}, \quad r_i > 0, \ j = 1, 2. \tag{2}$$

The neural mass model of Wilson and Cowan (1972) characterizes the behavior of spatially localized neural populations via a lumped parameter description. This model contains an excitatory and an inhibitory neural population that are coupled together and are considered as a single "node". The Wilson–Cowan model is described by

$$\dot{x}_i = -\Lambda_i x_i + F_i(\Upsilon_i + \Xi_i x_i + I_{x_i}), \tag{3}$$

where $x_i = [x_{E_i}, x_{I_i}]^T \in \mathbb{R}^2$ is a stack vector of the average membrane potentials of the excitatory and inhibitory populations, x_{E_i} and x_{I_i}, respectively. The vector $I_{x_i} = [I_{E_i}, I_{I_i}]^T \in \mathbb{R}^2$ represents the exogenous inputs that include the input from neighboring populations and/or external inputs such as controller inputs. The matrices $\Lambda_i, \Upsilon_i, \Xi_i$ are determined by

$$\Lambda_i = \begin{bmatrix} \alpha_{E_i} & 0 \\ 0 & \alpha_{I_i} \end{bmatrix}, \qquad \Upsilon_i = \begin{bmatrix} \rho_{E_i} \\ \rho_{I_i} \end{bmatrix}, \qquad \Xi_i = \begin{bmatrix} a_i & -b_i \\ c_i & -d_i \end{bmatrix}, \tag{4}$$

where a_i, b_i, c_i, d_i are positive constants and referred to as synaptic gains. The nonlinear function $F_i(\theta) : \mathbb{R}^2 \to \mathbb{R}^2$ is described by

$$F_i(\theta) = [f_i(\theta_1), f_i(\theta_2)]^T. \tag{5}$$

In order to interconnect Wilson–Cowan oscillators, it is assumed that the excitatory neural population of one node is coupled to the excitatory neural population of another node. The same coupling configuration is assumed for connection between inhibitory populations in two distinct nodes. In other words, if a node i is coupled to a node j with coupling gain w_{ij}, then the excitatory neural population and inhibitory neural population in node i are coupled to the excitatory and inhibitory neural populations in node j with the coupling gains w_{ij} and $-w_{ij}$, respectively. We note that this assumption is somewhat restrictive as these two interconnections can have different coupling gains in general (Hoppensteadt & Izhikevich, 1997; Ueta & Chen, 2003). Although the interconnection between nodes was originally considered as a direct coupling, it has been proposed that diffusive coupling can be utilized to control oscillatory behaviors and, in particular, synchrony behavior of populations (Ueta & Chen, 2003).

Now, consider a network with N Wilson–Cowan Oscillators interconnected with diffusive coupling. In this case, the dynamics of each node is represented by

$$\dot{x}_i = -\Lambda x_i + F_i\left(\Upsilon_i + \Xi_i x_i + D_s \sum_{j=1}^N w_{ij}(x_j - x_i)\right), \tag{6}$$

where $D_s = \text{diag}(1, -1)$ due to assuming the interconnections are restricted to being excitatory–excitatory and inhibitory–inhibitory.