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Brief paper

On the positive output controllability of linear time invariant systems☆

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ABSTRACT

This paper considers the output controllability of autonomous linear control systems that are subject to non-negative input constraints. Based on the evaluation of the geometric properties of the system, necessary and sufficient conditions are proposed for the *positive output controllability* of continuous linear time invariant systems. To aid in the practical evaluation of positive output controllability, additional sufficient conditions are derived for which efficient numerical techniques exist. These conditions are evaluated over a set of numerical examples which support the theoretical results.

1. Introduction

The property of controllability, introduced by Kalman, Ho, and Narendra (1962), evaluates the ability of a dynamic system to have its state driven from any initial state to any final state in a finite amount of time. The aim of studying the controllability properties of a dynamic system is to determine if a controller can be applied to generate a desired state space behaviour. For linear time invariant (LTI) systems, necessary and sufficient conditions have been identified (Hespanha, 2009; Kalman et al., 1962). For nonlinear time invariant systems, linearisation has been used to obtain sufficient conditions for local controllability (Sastry, 1999). In addition, sufficient conditions have been proposed using Lie algebra for local controllability and/or the global controllability of some nonlinear systems (Sastry, 1999; Sussmann, 1987).

When constraints are imposed on either the system states or inputs, the effect of the constraints can alter the controllability conditions. This paper focuses on non-negative input constraints, which are motivated by engineered systems such as non-prehensile mechanisms (Lynch & Mason, 1999), cable robots (Lau, Oetomo, & Halgamuge, 2011; Oh & Agrawal, 2005), one way valves (Willems, Heemels, De Jager, & Stoorvogel, 2002) and the antivibration control of pendulum systems (Saperstone & Yorke, 1971). This class of constraints have therefore been widely investigated resulting in different necessary and sufficient conditions for controllability being identified for continuous (Brammer, 1972; Heymann & Stern, 1975; Saperstone & Yorke, 1971; Yoshida & Tanaka, 2007) and discrete (Evans & Murthy, 1977) LTI systems. Additionally, sufficient conditions for local positive controllability of nonlinear systems have been obtained (Brammer, 1972; Goodwine & Burdick, 1996).

It is worthwhile to note that controllability is defined for states instead of outputs. In most engineering applications, tasks are defined for outputs, whose dimension can be much lower than that of the state. One example is the control of a multi-link cable driven manipulator, where the task is typically defined in terms of end effector pose, rather than the joint positions and velocities which can define the system's state (Lau, Oetomo, & Halgamuge, 2013). Under such a situation, it is natural to consider output controllability (see for example, García-Planas & Domínguez-García, 2013, Kreindler & Sarachik, 1964 and references therein). In the evaluation of output controllability, necessary and sufficient conditions for LTI systems are well established (Ogata, 2010). For systems subject to non-negative input constraints, there are no known results that consider output controllability.

In this paper, positive output controllability is defined for continuous LTI systems. Necessary and sufficient conditions for positive output controllability are derived. To more efficiently verify positive output controllability, some geometric sufficient

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conditions are proposed. These conditions are shown to be necessary and sufficient for two dimensional systems. The conditions are evaluated on numerical examples to support the theoretical results.

2. Preliminaries

2.1. Notation

Denote the set of real numbers as $\mathbb{R}$, the complex numbers as $\mathbb{C}$, the square identity matrix with m rows as I_m and the zero matrix with m rows and n columns as $0_{(m\times n)}$. If the vector $\mathbf{x} = [x_1 \ \dots \ x_n]^T \in \mathbb{R}^n$ satisfies $x_i > 0$ ($x_i \geq 0$) for all $i \in \{1, \dots, n\}$, then $\mathbf{x}$ is said to be positive (non-negative) and is denoted by $\mathbf{x} > \mathbf{0}$ ($\mathbf{x} \geq \mathbf{0}$). For any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the inner product is denoted $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{y}^T\mathbf{x}$. For the vector $\mathbf{x} \in \mathbb{C}^n$, the complex conjugate is denoted $\bar{\mathbf{x}}$ and an orthogonal vector $\mathbf{x}^\perp$.

Let an unforced continuous LTI system be given by

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x_0} \in \mathbb{R}^n, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$. The unforced response of the system (1) is given by $\mathbf{x}(t) = e^{At}\mathbf{x_0}$. The eigenvalues of A are denoted by the set $\Lambda(A) = \Lambda^r(A) \cup \Lambda^c(A)$, where $\Lambda^r(A) \subseteq \mathbb{R}$ represents the i purely real eigenvalues and $\Lambda^c(A) \subseteq \mathbb{C}$ the remaining j eigenvalues such that $i + j = n$. Let the i real eigenvalues of $\Lambda^r(A)$ be defined such that the $k \leq i$ distinct real eigenvalues λ are arranged in the form $\lambda_1 > \cdots > \lambda_k \in \Lambda^r$ and let the $l \leq \frac{j}{2}$ distinct real component ρ_j of the complex eigenvalues be arranged in the set $R^c(A) \subset \mathbb{R}$ such that $\rho_1 \geq \cdots \geq \rho_l \in R^c$ for $j \in \{1, \dots, l\}$. The corresponding eigenvectors for the eigenvalue λ is given by $\varepsilon(\lambda)$ and the set of all eigenvectors for A is given by the eigenspace $\mathcal{E}(A)$.

2.2. Geometric cone theory

Definition 1. A set $\mathcal{X} \subseteq \mathbb{R}^n$ is said to be a *cone* if for all $\mathbf{x} \in \mathcal{X}$ and $\alpha \geq 0$, $\alpha\mathbf{x} \in \mathcal{X}$. The set is a *convex cone* if it is a cone and for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, $\mathbf{x} + \mathbf{y} \in \mathcal{X}$ (Luenberger, 1968).

Definition 2. The *extreme rays* of the cone $\mathcal{X}$ are the rays that cannot be expressed as a positive linear combination of other rays in $\mathcal{X}$ (Barker, 1981).

Remark 1. Extreme rays form a positive linearly independent set that can provide a description of $\mathcal{X}$. An alternative description of $\mathcal{X}$ can be provided using the matrix $G \in \mathbb{R}^{q\times n}$, where $\text{cone}(G) = \{\mathbf{x} \in \mathbb{R}^n \mid G\mathbf{x} \leq \mathbf{0}\}$. ○

Definition 3. Let $K = [\mathbf{k}_1 \ \dots \ \mathbf{k}_m] \in \mathbb{R}^{n\times m}$ where n and m are positive integers. The *image* (or span) of the matrix K is defined as the set $\text{Im}(K) := \left\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \sum_{i=1}^m \alpha_i\mathbf{k}_i,\ \alpha_i \in \mathbb{R}\right\}$. The *positive span* of the matrix K is defined as the set $\text{span}_+(K) := \left\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = \sum_{i=1}^m \alpha_i\mathbf{k}_i,\ \alpha_i \geq 0\right\}$.

Definition 4. Let $\mathcal{X} \subseteq \mathbb{R}^n$. The *negative polar cone* of the set $\mathcal{X}$, denoted $\mathcal{X}^-$, is the set of all $\mathbf{y} \in \mathbb{R}^n$ such that $\langle \mathbf{y}, \mathbf{x}\rangle \leq 0\ \forall\, \mathbf{x} \in \mathcal{X}$ (Luenberger, 1968).

The negative polar cone and the positive span of a matrix always form convex cones by Definition 1.

2.3. Positive invariance of cones

Let $A \in \mathbb{R}^{n\times n}$ be a given state matrix of (1) and let λ represent an eigenvalue of A. The following definitions hold for λ and A.

Definition 5. A cone $\mathcal{X}$ is *positively invariant* with respect to system (1) if $\forall\, t > 0$, $e^{At}\mathcal{X} \subseteq \mathcal{X}$ (Castelan & Hennet, 1991).

Definition 6. If a subspace $\mathcal{Y} \subseteq \mathbb{R}^n$ is positively invariant with respect to (1) then the subspace is said to be *A-invariant* and for all $\mathbf{x} \in \mathcal{Y}$, $A\mathbf{x} \in \mathcal{Y}$ (Hespanha, 2009).

Definition 7. The *operating subspace* $\mathcal{O}(\lambda)$ is the largest A-invariant subspace such that for all $\mathbf{x} \in \mathcal{O}(\lambda)$, there exists matrices $M(t), N(t) \in \mathbb{C}^{n\times n}$ such that $e^{At}\mathbf{x} = \left(e^{\lambda t}M(t) + e^{\bar{\lambda} t}N(t)\right)\mathbf{x}$.

Remark 2. The operational subspace $\mathcal{O}(\lambda)$ is equal to $\varepsilon(\lambda)$ if λ is real and $\text{rank}(\varepsilon(\lambda)) = m$, where m is the algebraic multiplicity of λ. If λ is complex, then $\mathcal{O}(\lambda)$ is the plane of oscillation and in the case of defective matrices it is given by the span of $\varepsilon(\lambda)$ and the generalised eigenvectors. ○

Definition 8. Let $\mathcal{T} \subseteq \mathbb{R}^n$ be a positively invariant cone with respect to (1). The $\mathcal{T}$-dominant eigenvalue is the eigenvalue $\lambda^*(A, \mathcal{T})$ with largest real component such that $\exists$ a positively invariant cone $\mathcal{T}_1 \subseteq (\mathcal{T} \cap \mathcal{O}(\lambda^*))$ with dimension greater than 0. The $\mathcal{T}$-dominant eigenvectors $\varepsilon^*(A, \mathcal{T})$ are the corresponding eigenvectors of λ^*. The $\mathcal{T}$-dominant eigencone $\eta(A, \mathcal{T})$ is given by the intersection $\mathcal{T}$ and the $\mathcal{T}$-dominant eigenvectors such that $\eta(A, \mathcal{T}) = \mathcal{T} \cap \varepsilon^*(A, \mathcal{T})$.

Definition 9. Let $P \in \mathbb{R}^{p\times n}$, where $p \leq n$, be a projection matrix. The set of P-dominant eigenvectors $\mathcal{W}(A, P)$ is given by the set $\mathcal{W}(A, P) := \left\{\varepsilon \in \mathcal{E}(A) \mid \exists \mathbf{v} \in \mathbb{R}^p \text{ s.t. } \varepsilon = \varepsilon^*(A, \mathcal{P}(P, \mathbf{v}))\right\}$, where $\mathcal{P}(P, \mathbf{v})$ is the smallest positively invariant cone containing $P^T\mathbf{v}$ and ε^* is as given in Definition 8.

Remark 3. For a given positively invariant cone $\mathcal{T}$, the $\mathcal{T}$-dominant eigenvectors correspond to the eigenvectors with largest corresponding eigenvalue that has a non-zero intersection of its operating subspace with $\mathcal{T}$. The P-dominant eigenvectors are then the set of all possible $\mathcal{T}$-dominant eigenvectors where $\mathcal{T} \subseteq \text{Im}(P^T)$. ○

Definition 10. A matrix $H \in \mathbb{R}^{q\times q}$ is Metzler if its off-diagonal terms are non-negative.

A cone $\mathcal{X} = \text{cone}(G)$ is positive invariant if it satisfies the following result from Castelan and Hennet (1991, Proposition 2.1).

Proposition 1. *Cone(G) is positively invariant for the unforced continuous LTI system (1) iff $\exists$ a Metzler matrix $H \in \mathbb{R}^{q\times q}$ such that $GA = HG$.*

3. Problem formulation

Let a continuous LTI system be described by the n dimensional state $\mathbf{x} \in \mathbb{R}^n$, m dimensional input $\mathbf{u} \in \mathbb{R}^m$, p dimensional output $\mathbf{y} \in \mathbb{R}^p$ and system dynamics

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \qquad \mathbf{x}(0) = \mathbf{x_0}, \tag{2}$$

$$\mathbf{y} = C\mathbf{x}, \tag{3}$$

where A, B and C are matrices with appropriate dimensions.

The controllability property of (2) identifies if the input can be used to drive the state from any initial state to any final state. Formally *controllability* is defined as follows:

Definition 11. A continuous LTI system (2) is controllable if $\exists$ a finite time τ, such that $\forall\, \mathbf{x_0}, \mathbf{x_f} \in \mathbb{R}^n$ and $T \geq \tau$, $\exists$ an input trajectory $\mathbf{u}(\cdot)$ such that $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(T) = \mathbf{x}_f$ (Sastry, 1999).