Automatica 71 (2016) 229–236

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

# An obstruction to solvability of the reach control problem using affine feedback[☆]

Miad Moarref, Melkior Ornik, Mireille E. Broucke[1]

*Department of Electrical and Computer Engineering, University of Toronto, Toronto ON, Canada M5S 3G4*

ARTICLE INFO

*Article history:*
Received 5 August 2015
Received in revised form
3 February 2016
Accepted 11 April 2016
Available online 31 May 2016

*Keywords:*
Reach control problem
Affine feedback
Dual cones
Cone conditions

ABSTRACT

This paper studies the reach control problem (RCP) using affine feedback on simplices. The contributions of this paper are threefold. First, we identify a new obstruction to solvability of the RCP using affine feedback and provide necessary and sufficient conditions for occurrence of such an obstruction. Second, for two-input systems, these conditions are formulated in terms of scalar linear inequalities. Third, computationally efficient necessary conditions are proposed for checking the obstruction for multi-input systems as feasibility programs in terms of linear inequalities. In contrast to the previous work in the literature, no assumption is imposed on the set of possible equilibria, so the results are applicable to the general RCP.



## 1. Introduction

This paper studies the reach control problem (RCP) using affine feedback on simplices. Given an affine system defined on a simplex $\mathcal{S}$, the objective in the RCP is to design a feedback controller such that the trajectories of the closed-loop system leave $\mathcal{S}$ in finite time through a prespecified facet, without first leaving it through other facets. The RCP has been the subject of a great deal of research due to its fundamental importance in controlling a subclass of hybrid systems known as piecewise affine systems (Bemporad, Ferrari-Trecate, & Morari, 2000; Habets, Collins, & van Schuppen, 2006; Rodrigues, 2004). Piecewise affine systems are state-based switched systems where each discrete mode has a corresponding continuous-time affine dynamics. The discrete modes correspond to polytopes in the state space. For piecewise affine systems, reach control is at each mode to design a controller that prevents transitions of the closed-loop system to undesired discrete modes, and guarantees transition to the prespecified desired mode. The RCP has found applications in different fields including biomolecular networks (Belta, Habets, & Kumar, 2002), robot motion planning (Belta, Isler, & Pappas, 2005), aircraft control (Belta & Habets, 2006), robotic manipulators (Martino & Broucke, 2014), and aggressive maneuvers of mechanical systems (Vukosavljev & Broucke, 2014).

The first approaches to solve the RCP in Habets and van Schuppen (2004) and Roszak and Broucke (2006) lead to either conservative sufficient conditions or bilinear inequalities that are NP-hard. It later became evident that the (polytopic) set of possible closed-loop equilibria in the simplex, $\mathcal{O}_{\mathcal{S}}$, plays a crucial role in solvability of the RCP. In particular, several computationally efficient controller synthesis methods were devised by imposing the assumption that $\mathcal{O}_{\mathcal{S}}$ is a *face* of $\mathcal{S}$ (Ashford & Broucke, 2013; Broucke, 2010; Broucke & Ganness, 2014). The results were extended to polytopes in Helwa and Broucke (2013) and Lin and Broucke (2011). In Lin and Broucke (2011), the problem was to find a triangulation of the polytope and an associated piecewise affine feedback to solve the RCP assuming the system has $n-1$ inputs. The goal of Helwa and Broucke (2013) was to extend the results of Broucke (2010) directly to polytopes by formulating the so-called *monotonic RCP*. While all these works regard the RCP, the specific problems solved and the approaches are very different from those of this paper. In this paper, we focus on a sub-problem of the RCP regarding the ability to assign a non-vanishing affine function on $\mathcal{O}_{\mathcal{S}}$. We use a numerical, optimization-inspired approach whereas the previous works exploit system structure to arrive at analytical conditions for solvability. Finally, a Lyapunov theory for the RCP based on so-called flow functions (the analog of Lyapunov functions for stability analysis) was presented in Helwa and Broucke (2015). In this work, we do not assume existence of

[☆] The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Denis Arzelier under the direction of Editor Richard Middleton.

*E-mail addresses:* miad.moarref@utoronto.ca (M. Moarref), melkior.ornik@scg.utoronto.ca (M. Ornik), broucke@control.utoronto.ca (M.E. Broucke).

[1] Tel.: +1 416 978 0803; fax: +1 416 978 0804.

http://dx.doi.org/10.1016/j.automatica.2016.04.033

a flow function, and we are interested in necessary conditions for solvability rather than the analysis of a given controller.

More closely related to this work, recent research has focused on the existence and structure of the equilibria in the RCP (Semsar-Kazerooni & Broucke, 2014). The notion of reach controllability was introduced to characterize when closed-loop equilibria could be pushed off the simplex using affine feedback. Notions of topological and affine obstructions to solvability arose as necessary conditions to the solvability of RCP (Ornik & Broucke, 2015). The term "obstruction" is used in a similar spirit as in homotopy theory—to extend a continuous (or affine) map on a simplicial complex. The affine obstruction was studied in Ornik and Broucke (2015) for the case of two- and three-dimensional systems.

The main contributions of this paper are threefold. First, we formulate necessary and sufficient conditions for existence of a non-vanishing affine extension on $\mathcal{O}_{\mathcal{S}}$. To the best of our knowledge, this is the first result in the literature to present an obstruction to solvability of the RCP using affine feedback for multi-input systems and for the most general form of $\mathcal{O}_{\mathcal{S}}$. Second, we propose graphically motivated and computationally efficient necessary and sufficient conditions for checking the obstruction on $\mathcal{O}_{\mathcal{S}}$ for two-input systems in terms of scalar linear inequalities. Finally, computationally efficient necessary conditions are proposed for checking the obstruction on $\mathcal{O}_{\mathcal{S}}$ for multi-input systems as feasibility programs in terms of linear inequalities.

## 2. Problem formulation

Consider an $n$-dimensional simplex $\mathcal{S} := \mathrm{co}\{v_0, \ldots, v_n\}$, where $v_0, \ldots, v_n$ are $n+1$ affinely independent points in $\mathbb{R}^n$. Without loss of generality (w.l.o.g.) we assume $v_0 = 0$. Define $V_{\mathcal{S}} := \{v_0, \ldots, v_n\}$ to be the vertex set of $\mathcal{S}$. Let $\mathcal{F}_0, \ldots, \mathcal{F}_n$ denote the facets of $\mathcal{S}$, where each facet is indexed by the vertex it does not contain. We call $\mathcal{F}_0$ the *exit facet*. Let $h_j$, $j \in \{0, \ldots, n\}$, be the unit normal vector of facet $\mathcal{F}_j$ pointing outside of the simplex. Let $\mathbf{0}$ denote the singleton set $\{0\}$. Define $I := \{1, \ldots, n\}$ and let $I(x)$ be the minimal index set among $\{0, \ldots, n\}$ such that $x \in \mathrm{co}\{v_i \mid i \in I(x)\}$.

We consider an affine control system on $\mathcal{S}$ defined as

$$\dot{x} = Ax + Bu + a, \qquad x \in \mathcal{S}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times m}$, and $\mathrm{rank}(B) = m$. Define $\mathcal{B} := \mathrm{Im}(B)$, the image of $B$. Let $\phi_u(t, x_0)$ denote the trajectory of (1) starting at $x_0 \in \mathcal{S}$, under control input $u$, and evaluated at time instant $t$. Reach control theory studies the reachability of the exit facet $\mathcal{F}_0$ from any initial point in $\mathcal{S}$.

**Reach control problem (RCP).** Consider the affine system (1) defined on a simplex $\mathcal{S}$. Find an affine feedback $u(x) := Kx + g$, where $K \in \mathbb{R}^{m\times n}$ and $g \in \mathbb{R}^m$, such that for each $x_0 \in \mathcal{S}$ there exist $T \geq 0$ and $\delta > 0$ such that

(i) $\phi_u(t, x_0) \in \mathcal{S}, \forall t \in [0, T]$,
(ii) $\phi_u(T, x_0) \in \mathcal{F}_0$, and
(iii) $\phi_u(t, x_0) \notin \mathcal{S}, \forall t \in (T, T+\delta)$.

Two necessary conditions for solvability of the RCP by affine feedback are known (Habets & van Schuppen, 2004; Roszak & Broucke, 2006). First, the velocity vector $Ax + Bu(x) + a$ must point inside the cone generated by $\mathcal{S}$ at points in the facets $\mathcal{F}_i$, $i \in I$. This requirement is known as the *invariance conditions* (Roszak & Broucke, 2006). For $x \in \mathcal{S}$, define the closed, convex cone

$$\mathcal{C}(x) := \{y \in \mathbb{R}^n \mid h_j \cdot y \leq 0, \; j \in I \setminus I(x)\}. \tag{2}$$

Note that $h_0$ never appears in (2) and $\mathcal{C}(x) = \mathbb{R}^n$ for $x \in \mathcal{S}^\circ$, where $\mathcal{S}^\circ$ represents the interior of $\mathcal{S}$. Fig. 1 illustrates the cones $\mathcal{C}(v_i)$, $i \in \{0, 1, 2\}$, attached at the corresponding vertex $v_i$ to describe allowable directions for the vector field at the vertices.

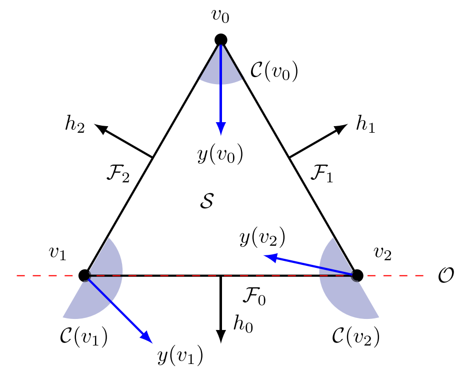


**Fig. 1.** A simplex $\mathcal{S} = \mathrm{co}\{v_0, v_1, v_2\}$ with vertices $V_{\mathcal{S}} = \{v_0, v_1, v_2\}$ and facets $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_2$. The facet $\mathcal{F}_i$, $i \in \{0, 1, 2\}$, is the convex hull of all vertices not including $v_i$. For each facet $\mathcal{F}_i$, the unit normal vector pointing out of $\mathcal{S}$ is shown by $h_i$. The cones $\mathcal{C}(v_i)$ are illustrated attached at each $v_i$ along with sample vectors $y_i \in \mathcal{C}(v_i)$.

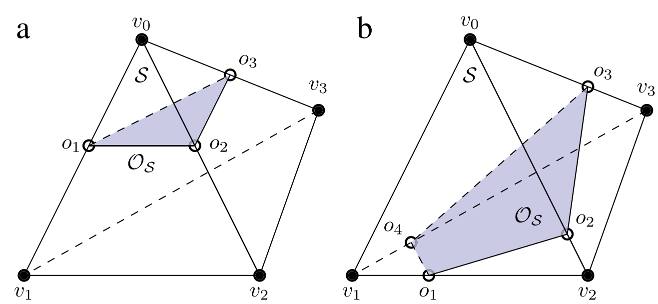


**Fig. 2.** Two hypothetical scenarios in $\mathbb{R}^3$ with $\mathcal{S} = \mathrm{co}\{v_0, v_1, v_2, v_3\}$; (a) $\mathcal{O}_{\mathcal{S}} = \mathrm{co}\{o_1, o_2, o_3\}$ is a simplex, and (b) $\mathcal{O}_{\mathcal{S}} = \mathrm{co}\{o_1, o_2, o_3, o_4\}$ is a polytope but not a simplex.

Here, e.g., since $I(v_0) = \{0\}$, we have $\mathcal{C}(v_0) = \{y \in \mathbb{R}^2 \mid h_j \cdot y \leq 0, \; j \in \{1, 2\}\}$. We say that $u(x)$ satisfies the invariance conditions if

$$Ax + Bu(x) + a \in \mathcal{C}(x), \quad \forall x \in \mathcal{S}. \tag{3}$$

A second necessary condition for the feedback $u(x)$ to solve the RCP is that there are no closed-loop equilibria in $\mathcal{S}$, i.e., $Ax + Bu(x) + a \neq 0$, for all $x \in \mathcal{S}$. It was shown in Habets et al. (2006) and Roszak and Broucke (2006) that these two necessary conditions combined form a sufficient condition for solvability of RCP using affine feedback. Closed-loop equilibria of (1) can only appear in the affine space

$$\mathcal{O} := \{x \in \mathbb{R}^n \mid Ax + a \in \mathcal{B}\}. \tag{4}$$

Therefore, we are interested in the feedback $u(x)$ that denies any equilibria in the set

$$\mathcal{O}_{\mathcal{S}} := \mathcal{S} \cap \mathcal{O} = \mathrm{co}\{o_1, \ldots, o_\kappa\}.$$

The intersection of a simplex $\mathcal{S}$ and an affine space $\mathcal{O}$ is either an empty set or a $\hat{\kappa}$-dimensional (compact and convex) polytope, where $0 \leq \hat{\kappa} \leq n$ and $\hat{\kappa} < \kappa$. We note that $\dim(\mathcal{O}) \geq m$. However, as $\mathcal{O}_{\mathcal{S}}$ might not pass through the interior of $\mathcal{S}$, there is no guarantee that $\dim(\mathcal{O}_{\mathcal{S}}) \geq m$. We define $V_{\mathcal{O}_{\mathcal{S}}} := \{o_1, \ldots, o_\kappa\}$ to be the set of vertices of $\mathcal{O}_{\mathcal{S}}$. Two examples of the set $\mathcal{O}_{\mathcal{S}}$ are shown in Fig. 2. Many papers in the literature study the RCP under the assumption that $\mathcal{O}_{\mathcal{S}}$ is a $\hat{\kappa}$-dimensional *face* of $S$. Due to the critical role of the set $\mathcal{O}_{\mathcal{S}}$ in the second necessary condition, relaxing the above assumptions and characterizing $\mathcal{O}_{\mathcal{S}}$ serve as major stepping stones for solving the RCP. In this paper, we introduce an obstruction to the RCP on the set of possible equilibria $\mathcal{O}_{\mathcal{S}}$ and, in contrast to the above mentioned papers, study $\mathcal{O}_{\mathcal{S}}$ in its most general form.

For all $x \in \mathcal{O}_{\mathcal{S}}$, the closed-loop vector field satisfies $Ax + Bu(x) + a \in \mathcal{B}$. Therefore, since $\mathcal{O}_{\mathcal{S}} \subseteq \mathcal{S}$, the existence of an affine map