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Brief paper

Coherent observers for linear quantum stochastic systems[☆]

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ABSTRACT

The theory of observers is a basic part of classical linear system theory. The purpose of this paper is to develop a theory of coherent observers for linear quantum systems. We provide a class of coherent quantum observers, which track the observables of a linear quantum stochastic system in the sense of mean values, independent of any additional quantum noise in the observer. We prove that there always exists such a coherent quantum observer described by quantum stochastic differential equations in the Heisenberg picture, and show how it can be designed to be consistent with the laws of quantum mechanics. We also find a lower bound for the mean squared estimation error due to the uncertainty principle. In addition, we explore the quantum correlations between a linear quantum plant and the corresponding coherent observer. It is shown that considering a joint plant–observer Gaussian quantum system, entanglement can be generated under the condition that appropriate coefficients of the coherent quantum observer are chosen, and this issue is illustrated in an example. These results pave the way towards observer-based quantum control.

1. Introduction

Effective control requires that sufficient information about the target plant is available to the controller. However, in many situations, full knowledge of the plant is not accessible and unknown quantities may be estimated on the basis of the available information (Ellis, 2002; Yamamoto, 2006). It is well established classically that the Kalman filter, which computes the conditional expectations of the state variables of the plant, is a statistical approach to state estimation based on dynamical linear-Gaussian models (Anderson & Moore, 1979). Likewise, in the emerging field of coherent quantum control, information on the quantum plant is also needed for efficacious control (Wiseman & Milburn, 2010). When the quantum plant is continuously monitored, the Belavkin–Carmichael quantum filter may be used to compute conditional expectations of plant variables (Belavkin, 1994; Bouten, van Handel, & James, 2007; Wiseman & Milburn, 2010). Mathematically, the quantum filter computes a quantum conditional expectation onto a *commutative* subspace of output signals generated by measurement processes. In the special case of linear quantum stochastic systems, i.e. those for which certain conjugate variables evolve linearly in the Heisenberg picture (as in linear quantum optics; e.g., see Wiseman & Milburn, 2010, James, Nurdin, & Petersen, 2008), the quantum filter reduces to the Kalman filter.

As a step towards better understanding of fully quantum *non-commutative* estimation and control, in this paper we extend Luenberger's approach (Luenberger, 1966) for observer design to linear quantum stochastic systems, whose dynamics are described by linear quantum stochastic differential equations (QSDEs). In classical linear systems theory, an observer is a system driven by the plant outputs (and control inputs where appropriate) and designed in such a way that the observer variables asymptotically track the plant variables. As is well known, if the plant is detectable, then the observer gain may be chosen so that the estimation error exponentially converges to zero.

In the quantum case we consider, as Fig. 1 shows, the output of the quantum plant is a fully quantum signal (e.g. an optical field), and no measurement is involved in this framework. This signal drives the *observer*, another fully quantum system which tracks

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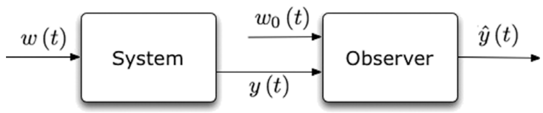

Fig. 1. A plant–observer composite quantum system.

the quantum plant asymptotically in the sense of mean values, in a series arrangement (Carmichael, 1993; Gough & James, 2009). Since the algebraic equations corresponding to a direct analog of the classical Luenberger observer need not correspond to a quantum physical system, here we have to consider the existence of such an observer constrained by physical realisability conditions distinguished from the classical case. We find explicit forms for the coefficients of a physically realisable coherent observer, and prove that there always exists a coherent quantum observer. This may involve including additional quantum noise in the observer (James et al., 2008; Miao & James, 2012). It is worth noting that the quantum observer we propose in this paper "observes" the observables of a linear quantum plant coherently, but not the quantum states. Here we mention another publication on quantum observers (Yamamoto, 2006), where the observers considered are classical systems, and measurement is involved.

In this paper, we give a full and general theorem with detailed proofs concerning the existence of coherent quantum observers, whereas only a partial result is presented in Miao and James (2012). New results for a simplified observer design for annihilation-operator linear quantum systems are included, and we show an explicit structure for the observer in this situation without additional quantum noise. Furthermore, we provide the fundamental limit for the mean squared estimation error by using a coherent observer. Observations in our work regarding quantum correlations in joint plant–observer quantum systems suggest there are inherent differences between tracking systems with classical and quantum components, making this topic not only practical but also interesting. Applications of coherent quantum observers can be expected, as the notion of an observer and estimate that we use has a history of success in the classical literature (Luenberger, 1966). We have applied the observer theory to fully quantum coherent tracking feedback control in Miao, James, and Ugrinovskii (2015). Moreover, quantum observers design using stronger criteria involving covariances have been developed in Miao, Hush, and James (2015).

The paper is organised as follows. Section 2 presents the linear quantum stochastic systems of interest as well as the physical realisability conditions. In Section 3, we give a detailed proof of the existence of coherent quantum observers, and provide a lower bound for the mean squared observer error. This is followed by Section 4, in which we analyse the quantum correlations in a joint plant–observer Gaussian system, including quantum discord. Section 5 provides some concluding remarks and future research directions.

Notation. In this paper, the asterisk is used to indicate the Hilbert space adjoint $X^\star$ of an operator X, as well as the complex conjugate $z^\star = x - iy$ of a complex number $z = x + iy$ (here, $i = \sqrt{-1}$ and $x, y \in \mathbb{R}$). The conjugate transpose $A^\dagger$ of a matrix $A = \{a_{ij}\}$ is defined by $A^\dagger = \{a_{ji}^\star\}$. Also defined are the conjugate $A^\sharp = \{a_{ij}^\star\}$ and the transpose $A^T = \{a_{ji}\}$ matrices, so that $A^\dagger = (A^T)^\sharp = (A^\sharp)^T$. Real and imaginary parts of a matrix A are denoted by $\Re(A)$ and $\Im(A)$ respectively. The mean value (quantum expectation) of an operator X is denoted by $\langle X \rangle$. The commutator of two operators X, Y is defined by $[X, Y] = XY - YX$. The anti-commutator of two operators X, Y is defined by $\{X, Y\} = XY + YX$. The tensor product of operators X, Y defined on Hilbert spaces H, G is denoted $X \otimes Y$, and is defined on the tensor product Hilbert space $\mathsf{H} \otimes \mathsf{G}$. I_n $(n \in \mathbb{N})$ denotes the n dimensional identity matrix. 0_n $(n \in \mathbb{N})$ denotes the n dimensional zero matrix.

2. Linear quantum stochastic systems

The dynamics of an open quantum system are uniquely determined by the triple $(S, L, \mathcal{H})$ (Gough & James, 2009; Parthasarathy, 1992). The self-adjoint operator $\mathcal{H}$ is the Hamiltonian describing the self-energy of the system. The unitary matrix S is a scattering matrix, and the column vector L with operator entries is a coupling vector. S and L together specify the interface between the system and the fields. Given an operator X defined on the initial Hilbert space H, its Heisenberg evolution is defined by

$$dX = (\mathcal{L}(X) - i[X, \mathcal{H}])\,dt + dW^\dagger S^\dagger [X, L] + \left[L^\dagger, X\right] S dW + \mathrm{tr}\left[\left(S^\dagger X S - X\right) d\Lambda_w\right], \quad (1)$$

with

$$\mathcal{L}(X) = \frac{1}{2} L^\dagger [X, L] + \frac{1}{2}\left[L^\dagger, X\right] L, \quad (2)$$

which is called the Lindblad superoperator. The operators W are defined on a particular Hilbert space called a Fock space F. When the fields (the number of fields is n_w) are in the vacuum states, these are the quantum Wiener processes which satisfy the Itô rule

$$dW dW^\dagger = I_{n_w} dt.$$

Input field quadratures $W + W^\sharp$ and $-i\left(W - W^\sharp\right)$ are each equivalent to classical Wiener processes, but do not commute. A field quadrature can be measured using homodyne detection (Gardiner & Zoller, 2000; Gough & James, 2009). The gauge processes Λ_w are input signals to the system as well.

We assume there is no interaction between different fields, and thus hereafter we assume S to be the identity matrix without loss of generality (James et al., 2008). This assumption eliminates the first term on the right hand side of (1). To be specific,

$$dX = (\mathcal{L}(X) - i[X, \mathcal{H}])\,dt + \frac{1}{2}\left([X, L] - [X, L^\dagger]\right) dW_1 - \frac{i}{2}\left([X, L] + [X, L^\dagger]\right) dW_2, \quad (3)$$

with

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} W + W^\sharp \\ -i(W - W^\sharp) \end{bmatrix}.$$

The quadrature form of the output fields is given by

$$\begin{bmatrix} dY_1 \\ dY_2 \end{bmatrix} = \begin{bmatrix} L + L^\sharp \\ -i(L - L^\sharp) \end{bmatrix} dt + \begin{bmatrix} dW_1 \\ dW_2 \end{bmatrix}. \quad (4)$$

In this work we focus on *open harmonic oscillators*. The dynamics of each oscillator are described by two self-adjoint operators position q_j and momentum p_k, which satisfy the canonical commutation relations $[q_j, p_k] = 2i\delta_{jk}$ where δ_{jk} is the Kronecker delta. It is convenient to collect the position and momentum operators of the oscillators into an n_x-dimensional column vector $x(t)$, defined by $x(t) = \left(q_1(t), p_1(t), q_2(t), p_2(t), \ldots, q_{\frac{n_x}{2}}(t), p_{\frac{n_x}{2}}(t)\right)^T$. In this case the commutation relations can be re-written as:

$$x(t)\,x(t)^T - \left(x(t)\,x(t)^T\right)^T = 2i\Theta_{n_x} \quad (5)$$

where $\Theta_{n_x} = I_{\frac{n_x}{2}} \otimes J$ with $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. In general, $\Theta_n = I_{\frac{n}{2}} \otimes J$ for any even number $n \in \mathbb{N}$.

Harmonic oscillators, in particular, are defined by having a quadratic Hamiltonian of the form $\mathcal{H} = \frac{1}{2} x^T R x$ with R being a $\mathbb{R}^{n_x \times n_x}$ symmetric matrix, and a coupling operator of the form $L = \Xi x$ with Ξ being a $\mathbb{C}^{\frac{n_w}{2} \times n_x}$ matrix (here n_x, n_w and n_y are positive even numbers). A special property of open harmonic oscillators