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Brief paper

Razumikhin and Krasovskii stability theorems for time-varying time-delay systems☆

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ABSTRACT

The main results of the paper are generalizations of the Razumikhin and of the Krasovskii classical stability theorems for stability analysis of time-varying time-delay systems. The condition of negativity of the time-derivative of Razumikhin functions and Krasovskii functionals is weakened. This is achieved by using the notion and properties of uniformly stable functions. We also show how to apply the results to the stability analysis of linear time-varying time-delay systems of retarded type. Both the system matrices and time-delays are allowed to be time-varying. Some constructive sufficient stability conditions are obtained and their effectiveness is demonstrated by some examples.

1. Introduction

It is well known that there are two ways to extend the Lyapunov direct method to time-delay systems. The first way is the Razumikhin approach, the second one is the Krasovskii approach (Hale, 1977). Both of them have been successfully used for the stability analysis and stabilization of time-invariant time-delay systems (see, for example, Hale, 1977; Kharitonov, 2013; Kojima, Uchida, Shimemura, & Ishijima, 1994; Zhang, Liu, & Feng, 2013), impulse time-delay systems (Chen & Zheng, 2011), functional difference systems (Melchor-Aguilar, Kharitonov, & Lozano, 2010; Pepe, 2014), and also allow to obtain some sufficient stability conditions for time-delay systems with time-varying coefficients and/or time-varying delays (see, for instance, Cacace, Conte, & Germani, 2016; Egorov & Mondié, 2015; Fridman & Orlov, 2009; Gu, Kharitonov, & Chen, 2003; Kharitonov & Niculescu, 2003; Mazenc & Malisoff, 2016). To guarantee asymptotic stability of a system by the Lyapunov–Krasovskii theorem we have to find a positive definite functional with a negative definite time derivative along the solutions of the system. By the Razumikhin theorem we need a positive definite function whose time-derivative is also negative definite under the Razumikhin condition (Hale, 1977).

The asymptotic stability analysis of time-varying system, for instance, for linear time-varying (LTV) system, is challenging. To see this, we notice that the stability analysis of LTV system has been listed as the first open problem in mathematical systems and control theory (see Aeyels & Peuteman, 1999). Even for a particular LTV system with periodic varying coefficients, the analysis and design are far from trivial, and thus some fundamental problems such as controllability (Savkin, 1997), stability analysis (Zhou, 2016) and stabilization (Savkin & Petersen, 2000), have received considerable attention in the literature (see Rugh, 1996; Zhou, 2016 and the references therein). In recent years there has been an increasing interest on the analysis and control of LTV systems with (time-varying) delays. For example, the classical predictor feedback for linear systems with constant delays has been extended to systems with time-varying delays and/or time-varying coefficients in Krstic (2010), Mazenc and Malisoff (2016) and Mazenc, Malisoff, and Niculescu (2014, 2015), and a finite dimensional predictor feedback referred to as truncated predictor feedback initially proposed in Zhou, Lin, and Duan (2012) has been extended to LTV systems with (time-varying) delays in Zhou (2014).

However, as was shown in Zhou (2016) for LTV systems described by delay free differential equations, the condition of negativity of the time-derivative of the Lyapunov function is quite

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conservative and should be weakened. This is also the case for time-delay systems. An intuitive understanding is this: a scalar LTV (time-delay) system can still be asymptotically stable even if it is unstable in a (short) period within which a time-invariant quadratic Lyapunov function(al), whose time-derivative is strictly decreasing, may not exist. To overcome such a shortcoming, one may use the results in Malisoff and Mazenc (2009) to relax the condition of negativity of the time-derivative of Lyapunov functions. Very recently, by using the notion of stable functions, it was able to weaken the condition of negativity of the time-derivative of the Lyapunov function in Zhou (2016) for delay free LTV system (one of the main results related with this paper is cited as Lemma 5 later).

In this contribution, motivated by the first author's work (Zhou, 2016), we show how to relax the condition of negativity of the time-derivative of Lyapunov function(al)s for time-varying time-delay systems. The main results of the paper are generalizations of the Razumikhin and Krasovskii stability theorems to the time-varying setting. In the proposed generalized Razumikhin and Krasovskii stability theorems, we no longer require that the time-derivatives of the Razumikhin functions and the Krasovskii functionals are negative definite for all t. It is shown that the obtained theorems allow us to deduce some constructive delay-independent and delay-dependent stability conditions for LTV systems with both time-varying coefficients and time-delays.

Another motivating paper for our research is Mazenc et al. (2015), where some stability conditions for LTV time-delay systems were obtained. The idea of the approach in Mazenc et al. (2015) is to extend the Halanay lemma (Halanay, 1966) to the time-varying case. The proposed conditions do not impose any constraints on the maximal value of the system parameters such as the delay and the norms of system coefficients. Instead, these are with some constraints on the integrals of the system parameters (functions). As noticed in Mazenc et al. (2015), these results cannot be obtained by using the classical Razumikhin and Krasovskii approaches. In this paper we show by examples that our proposed new Razumikhin and Krasovskii stability theorems can obtain some similar stability conditions that are less conservative than the results of Mazenc et al. (2015). To the best of our knowledge, for the proposed Razumikhin theorem, the only comparable result allowing indefinite time-derivatives of Razumikhin functions was given in Ning, He, Wu, and She (2014), which generalizes the results (Ning, He, Wu, Liu, & She, 2012) to time-delay systems. It will be clear that the result is more conservative than the one presented in our paper (see Remark 3 given later for details).

The paper is divided into five sections. Section 2 is devoted to the stability analysis of general nonlinear time-varying time-delay systems of retarded type by providing generalized Razumikhin and Krasovskii theorems. In Section 3 we show how to apply the results to the stability analysis of linear systems with both time-varying coefficients and time-varying delays. In Section 4 we give three numerical examples. Section 5 concludes the paper.

2. Razumikhin and Krasovskii stability theorems

2.1. Preliminaries and motivation

Throughout this paper, if not specified, the symbol J denotes the set $[t^{\#}, \infty)$ with $t^{\#}$ being some constant. Consider the time-varying (TV) time-delay system

$$\dot{x}(t) = f(t, x_t), \quad x_{t_0} = \phi, \; t, \; t_0 \in J, \; t \geq t_0, \tag{1}$$

where $f(t, \phi) : J \times \mathbb{C}([-\tau, 0], \mathbf{R}^n) \to \mathbf{R}^n$ is a known function that is piecewise continuous with respect to t, locally Lipschitz in ϕ, and $f(t, 0) = 0$. The restriction of the solution $x(t)$ of system (1) to the interval $[t - \tau, t]$ is denoted by $x_t : \theta \to x(t + \theta), \theta \in [-\tau, 0]$, where τ is the delay. In this paper, $\mathbb{PC}(\mathcal{L}, \mathbf{R}^n)$ is the space of $\mathbf{R}^n$-valued piecewise continuous functions defined on $\mathcal{L}$, and $J_\tau = [t^{\#} - \tau, \infty)$.

The TV time-delay system (1) is said to be *globally uniformly asymptotically stable* (GUAS), if there exists a $\mathcal{KL}$-function κ such that

$$|x(t)| \leq \kappa\left(\left\|x_{t_0}\right\|, t - t_0\right), \quad t, \; t_0 \in J, \; t \geq t_0,$$

where $|\cdot|$ refers to the usual Euclidean norm and $\|x_t\| = \sup_{s\in[-\tau, 0]} |x(t + s)|$. It is said to be *globally uniformly exponentially stable* (GUES) if there exist two constants $\alpha > 0$ and $\beta > 0$ such that

$$|x(t)| \leq \beta e^{-\alpha(t-t_0)} \left\|x_{t_0}\right\|, \quad t, \; t_0 \in J, \; t \geq t_0.$$

The following two theorems are well known.

Lemma 1 (*Razumikhin Theorem, Th. 4.2 in Hale, 1977*)**.** *The TV time-delay system (1) is GUAS if there exist a continuous function $V(t, x)$, $t \in J_\tau$, $x \in \mathbf{R}^n$, three $\mathcal{K}_\infty$-functions u, v, w and a continuous nondecreasing function $q(s) > s$ for $s > 0$, such that the following two conditions are met for all $t \in J$:*

$$\begin{aligned} &u(|x|) \leq V(t, x) \leq v(|x|), \quad x \in \mathbf{R}^n, \\ &\dot{V}(t, x(t)) \leq -w(|x(t)|) \\ &\text{if } V(t + s, x(t + s)) \leq q(V(t, x(t))), \quad \forall s \in [-\tau, 0]. \end{aligned} \tag{2}$$

Lemma 2 (*Krasovskii Theorem, Th. 2.1 in Hale, 1977*)**.** *The TV time-delay system (1) is GUAS if there exist a continuous functional $V(t, \phi)$, $t \in J$, $\phi \in \mathbb{PC}([-\tau, 0], \mathbf{R}^n)$, and three $\mathcal{K}_\infty$-functions u, v, w such that the following two conditions are met for all $t \in J$:*

$$\begin{aligned} &u(|\phi(0)|) \leq V(t, \phi) \leq v(\|\phi\|), \quad \phi \in \mathbb{PC}([-\tau, 0], \mathbf{R}^n), \\ &\dot{V}(t, x_t) \leq -w(|x(t)|). \end{aligned}$$

We notice that in the Krasovskii stability theorem the time-derivative of the Krasovskii functional $V(t, x_t)$ is required to be negative definite, and in the Razumikhin stability theorem the time-derivative of the Razumikhin function $V(t, x(t))$ is also required to be negative definite under the Razumikhin condition $V(t + s, x(t + s)) \leq q(V(t, x(t))), \forall s \in [-\tau, 0]$. As noticed in Mazenc et al. (2015), these conditions are either very strong or difficult to meet in practice (this is also the case for delay free TV systems as pointed out in Zhou (2016)). In the following we use an example to demonstrate the situation.

Example 1. Consider the scalar time-delay system

$$\dot{x}(t) = -x(t) + b(t)x(t - h(t)), \tag{3}$$

where $b(t)$ and $h(t) \geq 0$ are scalar functions. Such a system has been previously studied in many references (see, for example, p. 129 in Hale (1977) and Mazenc et al. (2015)). Let $V(x(t)) = x^2(t)$. Then it was shown in Hale (1977) that, by using the classical Razumikhin theorem (Lemma 1), condition (2) is satisfied (or the system is GUAS) if $|b(t)| < 1$. Now consider a periodic function $b(t)$ with period $\omega = 1$, and

$$b(t) = \begin{cases} 0, & t \in [0, c), \\ d, & t \in [c, 1), \end{cases} \tag{4}$$

where $c \in (0, 1)$ and $d > 0$ is some constant (Mazenc et al., 2015). If the classical Razumikhin theorem is applied, it follows from the above discussion that the system is GUAS if

$$d < 1. \tag{5}$$