Automatica 71 (2016) 314–317

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

Asymptote angles of polynomic root loci ☆

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ARTICLE INFO

Article history:
Received 22 May 2015
Received in revised form 23 February 2016
Accepted 12 May 2016
Available online 10 June 2016

Keywords:
Polynomic root locus
Rouché's theorem
Continuous dependence
Geometry of the roots

ABSTRACT

In recent publications the plotting rules of polynomially parameterized root loci have been developed for characteristic polynomials in the gain as unknown with degrees 2 and 3. Unfortunately the proof of the asymptote rule is very long and intricate. In this note a generalization of the rule of asymptote angles is addressed for any polynomial with arbitrary degree. Additionally, a simple and shorter proof of the asymptote angles is obtained through the continuous dependence of the roots of the complex characteristic polynomial with respect to a parameter. The geometry of the roots of this polynomial is investigated via a variant of Rouché's theorem.

1. Introduction

The polynomic root locus (PRL) is the geometric locus of the roots of a polynomially parameterized characteristic polynomial. This polynomial appears in the denominator of a closed loop transfer function when the controller is rational in the parameter (usually a gain K). For example given a transfer function of a plant $G_p(s) = \frac{N(s)}{D(s)}$ where $N, D \in \mathbb{R}[s]$, and a controller $G_c(s) = \frac{K}{D_c(s;K)}$ with $D_c \in \mathbb{R}[s,K]$ being a bivariate polynomial, the characteristic polynomial $D_c(s;K)D(s)+KN(s)$ is polynomic in K. The motivation of nonaffine root loci stems from the fact that affine root loci (ARL) are not high-gain stabilizing for minimum-phase linear systems with relative degree greater than two. This means that controllers are unable to stabilize the closed-loop system for sufficiently large K since at least one closed-loop pole diverges to infinity through the right-half complex plane. Quadratic root loci (QRL) can high-gain stabilize minimum-phase linear systems with relative degree up to 3, cubic root loci (CRL) up to 4, and it is conjectured that higher-order root loci could potentially high-gain stabilize any minimum-phase linear system with known relative degree. The problem of high-parameter stabilizing for minimum-phase systems relies on the derivation of the angles of the asymptotes of the PRL.

Some applications of the PRL in control theory can be found in the direct adaptive stabilization of minimum-phase systems, and in the design of linear quadratic regulators, Hoagg and Bernstein (2007a,b) and Miller and Davison (1991).

In the ARL plotting the computation of the angles of the asymptotes stands for one of the most involved derivations. In the PRL the computation turns out to be more complicated since the PRL exhibits a wider variety of asymptotes. For the particular case of quadratically parameterized characteristic polynomials, the asymptote rules of the QRL were firstly stated by Géher and Kóta in the field of analysis and synthesis of active networks, Géher and Kóta (1967). They proposed a method based on the dependence of the gain on the complex variable through radicals of polynomials; unfortunately Géher and Kóta do not provide proofs of these rules, and the asymptote rules are inaccurate. Recently, the QRL has attracted the attention of some control researchers. In Wellman and Hoagg (2014), the asymptote rules of the QRL are derived by exploiting the Newton's binomial formula and making an asymptotic approximation of the geometric locus with the big O notation. The inconvenience of that approach is that the proof is very long (five pages), obscure and intricate. The CRL plotting rules are developed in Wellman (2012) for a particular characteristic polynomial p_c of the form

$$p_c(s;K) = p(s)\prod_{i=1}^{2}(s-\rho_i+\gamma_i K) + K^3 q(s)$$

with $\gamma_i, \rho_i \in \mathbb{R}$ and $p, q \in \mathbb{R}[s]$. As an attempt to extend and to simplify the proof of the asymptote rule it is worth noticing that the problem of determination of the asymptotes is actually framed in a general problem of continuous dependence of complex polynomials on their coefficients. This technical note follows this spirit and presents a simpler novel proof in which it is highlighted

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Tingshu Hu under the direction of Editor André L. Tits.

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http://dx.doi.org/10.1016/j.automatica.2016.05.025

that the limiting distribution of the far roots is a set of Butterworth patterns (roots of unit); this pattern is also common in the network theory for the design of filters and in control for the design of linear quadratic regulators.

This note is organized as follows. The background and notation are exposed in Section 2 which is followed by a simple derivation of the asymptote angles in Section 3. The concluding remarks are discussed in Section 4.

2. Notation and background

The cardinality of a set S is denoted as $\#(S)$. $\mathbb{C}$ is the field of complex numbers, $\mathbb{R}$ is the field of real numbers, $\mathbb{Q}$ is the field of rational numbers and $\mathbb{R}^n$ is the n-dimensional Euclidean space. The vectors of the standard basis in $\mathbb{R}^n$ are denoted as $\mathbf{e}_i$ with $1 \le i \le n$. The ring of polynomials over the field $\mathbb{R}$ in the unknown $s \in \mathbb{C}$ is represented as $\mathbb{R}[s]$; the degree of a polynomial $p \in \mathbb{R}[s]$ is indicated by $\deg p$. The polynomials in the unknown $s \in \mathbb{C}$ are represented by making explicit the vector of real coefficients, i.e. if $p \in \mathbb{R}[s]$ is such that $p(s) = \sum_{i=0}^{n} a_i s^i$ with $a_i \in \mathbb{R}$, then the notation $p(s;\mathbf{a})$ with $\mathbf{a} \in \mathbb{R}^{n+1}$ is adopted. When the coefficients of $p(s;\mathbf{a})$ depend on the parameter $K \in \mathbb{R}$, we write $p(s;\mathbf{a}(K))$ or $p(s;K)$. The imaginary number is $\mathbf{i} = \sqrt{-1}$. In the complex plane an open disk with center $s_0 \in \mathbb{C}$ and a radius $\epsilon > 0$ is represented as $\mathbf{D}(s_0,\epsilon)$; the closed disk is obtained by taking the closure of the open disk, i.e. $\overline{\mathbf{D}(s_0,\epsilon)}$. In the parameter space, an open ball of center $\tilde{\mathbf{a}} \in \mathbb{R}^{n+1}$ and radius $\delta > 0$ is defined as $B(\mathbf{a},\delta) \triangleq \left\{\tilde{\mathbf{a}} \in \mathbb{R}^{n+1} : \left\|\mathbf{a}-\tilde{\mathbf{a}}\right\| < \delta\right\}$.

Let $\{p_i \in \mathbb{R}[s] : i = 0,\ldots,m\}$ be a family of monic polynomials $p_i(s) = \sum_{j=0}^{n_i} a_j^{(i)} s^j$ with $n_i \ge n_{i-1} > 0$ for $i = 1,\ldots,m$. Given a parameter $K \ge 0$ and constants $\gamma_i > 0$ for $i = 0,\ldots,m$, let $p_c(s;K)$ be the following characteristic polynomial in the unknown $s \in \mathbb{C}$ and polynomially parameterized in K,

$$p_c(s;K) = \sum_{i=0}^{m} \gamma_i K^{m-i} p_i(s). \tag{1}$$

From the chain of degrees n_i it is clear that the degree of p_c as a polynomial in s is $\deg(p_c) = n_m$. It is common in control theory to select $n_m > n_{m-1}$ and $\gamma_m = 1$ to have a monic characteristic polynomial.

The PRL is defined as follows:

Definition 1. Given $p_c(s;K)$ as in (1), the polynomic root locus is the geometric locus of the complex plane $\Omega = \{z \in \mathbb{C} : p_c(z;K) = 0 \text{ with } K \in [0,\infty)\}$.

When $K = 0$, $p_c(s;0) = \gamma_m p_m(s)$ and the roots of p_c are those of p_m. For the asymptotic location of the roots of $p_c(s;K)$ it is convenient to divide $p_c(s;K)$ by K^m which results in the polynomial

$$P(s;K) = \sum_{i=0}^{m} \frac{\gamma_i}{K^i} p_i(s). \tag{2}$$

This polynomial has the same roots as $p_c(s;K)$. From the characteristic equation $P(s;K) = 0$, as $K \to \infty$ it is clear that n_0 roots of $P(s;K)$ tend to the roots of p_0 and the other $n_m - n_0$ roots follow asymptotes, i.e. lines that cut the branches of the root locus Ω at improper points in the infinity; formally the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ should be considered to take into account those roots at the infinity.

To make explicit the dependence of p_i on its coefficients, this polynomial will be written as $p_i(s;\mathbf{c}_i)$ with $\mathbf{c}_i \in \mathbb{R}^{n_i+1}$. Also for simplicity the coefficient vectors are subsumed in $\mathbb{R}^{n+1}$ by adding zero entries. Then the polynomial $P(s;K)$ can be considered as a perturbed polynomial from $\gamma_0 p_0(s;\mathbf{c}_0)$ via the coefficient perturbation $\mathbf{c} \mapsto \tilde{\mathbf{c}}(K) = \gamma_0\mathbf{c}_0 + \sum_{i=1}^{m} \frac{\gamma_i}{K^i}\mathbf{c}_i$. Then $P(s;K)$ is written as $P\left(s;\tilde{\mathbf{c}}(K)\right)$ and its roots will be denoted as $r_1(K),\ldots,r_n(K)$.

For the relative degrees between pairs of polynomials $\left(p_i,p_j\right)$ with $j > i$, the following notation is introduced: $d_i^j \triangleq \sum_{k=i+1}^{j} d_k = n_j - n_i$. For the sake of clarity, given a nonzero integer x, the notation $\bar{x}$ is used to describe the set $\{1,2,\ldots,x\}$.

3. Angles of the asymptotes of the PRL

This section is devoted to derive a proof of the asymptotes of the PRL based on the continuous dependence of the roots of a complex polynomial with respect to its coefficients.

The main result is based on the following theorem on the localization of zeros of complex polynomials. Basically, it is a reformulation of Rouché's theorem to deal with the continuous dependence of the roots of a polynomial on its coefficient vector, Bhattacharyya, Chapellat, and Keel (1995):

Theorem 2. *Let $P(s;\mathbf{a})$ be a non-constant polynomial of order n, and $z_1,\ldots,z_l$ its roots with multiplicities $\alpha_1,\ldots,\alpha_l$. For a given $\epsilon > 0$ such that the closed disks $\overline{\mathbf{D}\left(z_j,\epsilon\right)}$ are mutually disjoint, there exists a $\delta(\epsilon) > 0$ such that for all $\tilde{\mathbf{a}} \in B(\mathbf{a},\delta(\epsilon))$ it is verified that there exist exactly α_j roots of $P\left(s;\tilde{\mathbf{a}}\right)$ inside the disk $\mathbf{D}\left(z_j,\epsilon\right)$ and $n-\deg P\left(\cdot;\tilde{\mathbf{a}}\right)$ outside the disk $\mathbf{D}\left(0,\frac{1}{\epsilon}\right)$, counting multiplicities.*

A natural consequence of Theorem 2 is that the roots r_j of $P\left(s;\tilde{\mathbf{c}}(K)\right)$ can be represented as continuous functions of K.

The main contribution of this communication is the following proposition:

Proposition 3. *Let $P(s;\mathbf{c}(K))$ be the polynomial in (2) with roots $r_j(K)$ for $j = 1,2,\ldots,n_m$. Then n_0 roots $\left\{r_j(K) : j = 1,2,\ldots,n_0\right\}$ of $P(s;\mathbf{c}(K))$ tend to the roots of $p_0(s)$ as $K \to \infty$, while the remaining d_0^m roots $\left\{r_j(K) : j = n_0+1,\ldots,n_m\right\}$ have the following asymptotic tendency: Let us consider the set of indices $I_0 = \{i_1,i_2,\ldots,i_l\} \subseteq \{0,1,\ldots,m\}$ with a total order $i_1 < i_2 < \cdots < i_l$ and $l \ge 2$. For the configuration in the space of relative degrees $(\left(d_0^1,d_1^2,\ldots,d_{m-1}^m\right)$-space) given by*

$$(i-i_1)\, d_{i_1}^{i_2} = (i_2-i_1)\, d_{i_1}^{i} \quad \text{for } i \in I_0 \tag{3}$$

$$(i_2-i_1)\, d_{i_1}^{i} < (i-i_1)\, d_{i_1}^{i_2} \quad \text{for } i \in \{0,1,\ldots,m\} \setminus I_0$$

the roots of P asymptotically tend to clusters of asymptotes

$$\Omega_K^k = \left\{ \sqrt[d_{i_1}^{i_2}]{\left(|y_k|\,K\right)^{(i_2-i_1)}}\, e^{\mathbf{i}\frac{(i_2-i_1)(2\pi j+\arg(y_k))}{d_{i_1}^{i_2}}} : j \in \bar{d}_{i_1}^{i_2} \right\}$$

for $k = 1,\ldots,i_l-i_1$, where y_k is a solution of the algebraic equation: $\gamma_{i_1} + \gamma_{i_2} y^{(i_2-i_1)u} + \gamma_{i_3} y^{(i_3-i_1)u} + \cdots + \gamma_{i_l} y^{(i_l-i_1)u} = 0$ with $u = \frac{d_{i_1}^{i_2}}{i_2-i_1}$. The angles of the asymptotes are then $\frac{(i_2-i_1)(2\pi j+\arg(y_k))}{d_{i_1}^{i_2}}$ for $j \in \bar{d}_{i_1}^{i_2}$.

Proof. Let us assume that $p_0(s;\mathbf{c}_0)$ has l_0 roots z_j with multiplicities α_j for $j = 1,2,\ldots,l_0$. In virtue of Theorem 2, given a sufficiently small $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that each closed disk $\overline{\mathbf{D}\left(z_j,\epsilon\right)}$ exactly contains α_j roots of $P\left(s;\tilde{\mathbf{c}}(K)\right)$ for all $\tilde{\mathbf{c}}(K) \in \mathbf{D}(\gamma_0\mathbf{c}_0,\delta(\epsilon))$; recall that $\left\|\tilde{\mathbf{c}}(K) - \gamma_0\mathbf{c}_0\right\| = \frac{1}{|K|}\left\|\sum_{i=1}^{m} \frac{\gamma_i}{K^{i-1}}\mathbf{c}_i\right\|$ and $\lim_{K\to\infty} \tilde{\mathbf{c}}(K) = \gamma_0\mathbf{c}_0$. We can always find a constant $K_0 > 0$ sufficiently large such that for $|K| > K_0$, $\tilde{\mathbf{c}}(K)$ is guaranteed to be in the disk $\mathbf{D}(\gamma_0\mathbf{c}_0,\delta(\epsilon))$. Therefore n_0 roots of $P(s;\mathbf{c}(K))$ approach to the roots of $p_0(s;\mathbf{c}_0)$ as $K \to \infty$. The other d roots