Automatica 70 (2016) 158–172

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

A probabilistic interpretation of set-membership filtering: Application to polynomial systems through polytopic bounding[☆]

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ARTICLE INFO

Article history:
Received 5 May 2015
Received in revised form
7 January 2016
Accepted 6 March 2016
Available online 16 April 2016

Keywords:
State estimation
Filtering
Set-membership estimation
Set of probability measures
Sum-of-squares polynomials

ABSTRACT

Set-membership estimation is usually formulated in the context of set-valued calculus and no probabilistic calculations are necessary. In this paper, we show that set-membership estimation can be equivalently formulated in the probabilistic setting by employing sets of probability measures. Inference in set-membership estimation is thus carried out by computing expectations with respect to the updated set of probability measures $\mathcal{P}$ as in the probabilistic case. In particular, it is shown that inference can be performed by solving a particular semi-infinite linear programming problem, which is a special case of the truncated moment problem in which only the zeroth order moment is known (i.e., the support). By writing the dual of the above semi-infinite linear programming problem, it is shown that, if the nonlinearities in the measurement and process equations are polynomial and if the bounding sets for initial state, process and measurement noises are described by polynomial inequalities, then an approximation of this semi-infinite linear programming problem can efficiently be obtained by using the theory of sum-of-squares polynomial optimization. We then derive a smart greedy procedure to compute a polytopic outer-approximation of the true membership-set, by computing the minimum-volume polytope that outer-bounds the set that includes all the means computed with respect to $\mathcal{P}$.

1. Introduction

Inferring the value of the state of a dynamical system at the various time instants is a classical problem in control and estimation theory. The state is estimated based on noisy signal observations and on a state transition model, which in turn is affected by two sources of uncertainty (namely, process disturbance and uncertainty on the initial state conditions). In the literature, there are two main approaches for dealing with the uncertainties and noises acting on the system:

- the *stochastic (probabilistic) approach* that assumes that the noises and the uncertainties are unknown but they can be described by known probability distributions.
- the *set-membership approach* that assumes that the noises and the uncertainties are unknown but bounded in some compact sets.

The probabilistic approach is grounded on Bayesian filtering, whose aim is to update with the measurements and propagate up on time the *probability density function* (PDF) of the state. Inferences are then carried out by computing expectations with respect to this PDF, i.e., mean, variance, credible regions. It is well known that, for linear discrete-time dynamical systems corrupted by Gaussian noises, the Bayesian filter reduces to the Kalman filter.

The set-membership approach is instead based on the construction of a compact set which is guaranteed to include the state values of the system that are consistent with the measured output and the assumed bounds on the noises/disturbances (Casini, Garulli, & Vicino, 2014; Cerone, Piga, & Regruto, 2012a; Combettes, 1993; Milanese, Norton, Piet-Lahanier, & Walter, 1996; Milanese & Novara, 2011; Milanese & Vicino, 1991). This compact set is propagated in time and updated recursively with the output observations. In set-membership estimation, computing inferences thus means to determine this compact set. Set-membership estimation was first proposed in Bertsekas and Rhodes (1971) and Schweppe (1967),

[☆] The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Antonio Vicino under the direction of Editor Torsten Söderström.

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http://dx.doi.org/10.1016/j.automatica.2016.03.021

where an ellipsoidal bounding of the state of linear dynamical systems is computed. The application of ellipsoidal sets to the state estimation problem has also been studied by other authors, for example (Kuntsevich & Lychak, 1992; Savkin & Petersen, 1998), and, independently, in the communications and signal processing community, starting from the works (Deller & Luk, 1989; Deller, Nayeri, & Liu, 1994; Deller & Odeh, 1989; Fogel & Huang, 1982). In order to improve the estimation accuracy, the use of a convex polytope instead of an ellipsoid has been proposed in Mo and Norton (1990) and Piet-Lahanier and Walter (1989). Unfortunately such a polytope may be extremely complex and the corresponding polytopic updating algorithms may require an excessive amount of calculations and storage (without any approximations, the number of vertices of the polytope increases exponentially in time). For this reason, it has been suggested to outer approximate the true polytope with a simpler polytope, i.e. possessing a limited number of vertices or, equivalently, faces (Broman & Shensa, 1990). In this respect, a parallelotopic approximation of the set-membership set was presented in Chisci, Garulli, Vicino, and Zappa (1998) and Chisci, Garulli, and Zappa (1996). A parallelotope is the generalization of the parallelogram to $\mathbb{R}^n$. Minimum-volume bounding parallelotopes are then used to estimate the state of a discrete-linear dynamical system with polynomial complexity. Zonotopes have been proposed to reduce the conservativeness of parallelotopes. Intuitively zonotopes are polytopes with parallel faces, for a more precise definition see Le, Stoica, Alamo, Camacho, and Dumur (2013, chap. 2). A parallelotope is thus a special zonotope. Zonotopes are used in Combastel (2003), Le, Alamo, Camacho, Stoica, and Dumur (2011) and Puig, Saludes, and Quevedo (2003) to build a state bounding observer in the context of linear discrete systems.

Zonotopes are also employed to address the problem of set-membership estimation for non-linear discrete-time systems with a bounded description of noise and uncertainties (Alamo, Bravo, & Camacho, 2003). At each sample time, a guaranteed bound of the uncertain state trajectory of the system is calculated using interval arithmetic applied to the nonlinear functions through the mean interval extension theorem. This outer bound is represented by a zonotope. Similar approaches for set-membership estimation for nonlinear systems are presented in Calafiore (2005), El Ghaoui and Calafiore (2001) and Maier and Allgöwer (2009), where ellipsoids are used instead of zonotopes. Recently, randomized methods are used in Dabbene, Henrion, Lagoa, and Shcherbakov (2015) to approximate, with probabilistic guarantees, the uncertain state trajectory with polynomial sublevel sets.

The aim of this paper is to address the problem of the estimation of the state of a discrete-time non-linear dynamical system (characterized by polynomial non-linearities) in which initial state and noises are unknown but bounded by some compact sets (defined by polynomial inequalities). We are therefore in the context of set-membership estimation, but we will address this problem in a very different way from the approaches presented above. We reformulate set-membership in the probabilistic setting and solve it using the theory of moments and positive polynomials. More precisely the contributions are the following.

First, by exploiting recent results on filtering with sets of probability measures (Benavoli, 2013; Benavoli, Zaffalon, & Miranda, 2011), we show that set-membership estimation can be equivalently formulated in a probabilistic setting by employing sets of probability measures. In particular, we show that the prediction and updating steps of set-membership estimation can be obtained by applying Chapman–Kolmogorov equation and Bayes' rule point-wise to the elements of this set of probability measures $\mathcal{P}$. This unifies the probabilistic approach (Bayes filter) and the set-membership approach to state estimation. This result can have an enormous impact, because it finally can allow us to combine set-membership and classical probabilistic uncertainty in order to obtain hybrid filters, i.e., stochastic (probabilistic) filters that are for instance able to use information about the bounding region as well as the probabilistic moments (mean and variance) of the noises or that are able to deal with a Gaussian measurement noise and a bounded, with known moments, process noise, etc. Moreover, it can allow us to compute credible regions (Bayesian confidence intervals) that takes into account both deterministic and probabilistic uncertainty, as well as it allows us to make decisions by choosing the action that minimizes the expectation of some loss function (this is important, for instance, in control design). In the context of this paper a first attempt in combining deterministic and probabilistic uncertainty has been proposed in Benavoli (2013), while Combastel (2015) has proposed a joint Zonotopic and Gaussian Kalman filter for discrete-time LTV systems simultaneously subject to bounded disturbances and Gaussian noises. The work (Fernandez-Canti, Blesa, & Puig, 2013) instead proposes a Bayesian approach to set-membership estimation imposing a uniform distribution on the membership-set similar to the idea proposed in Gning, Mihaylova, and Abdallah (2010); Gning, Ristic, and Mihaylova (2011). We will show that this approach is different from set-membership estimation, since set-membership estimation cannot be interpreted in the Bayesian framework, but only in the framework of set of probability measures.

Second, under this probabilistic interpretation, inferences in set-membership estimation are carried out by computing expectations with respect to the set $\mathcal{P}$ as in the probabilistic case. In particular, we show that the membership set $\mathcal{X}$ (i.e., the set that includes the state with guarantee) can be obtained by computing the union of the supports of the probability measures in $\mathcal{P}$. Moreover, we prove that a minimum volume convex outer-approximation of $\mathcal{X}$ can simply be obtained by computing the set $\mathcal{M}$ that includes all the means computed with respect to the probabilities in $\mathcal{P}$. The proof is not constructive, hence we do not have a convenient description of $\mathcal{M}$. However we show that we can determine the least conservative half-space $\mathcal{H}$ that includes M, by solving a semi-infinite linear programming problem. This problem is a special case of the truncated moment problem (Krein & Nudelḿan, 1977; Lasserre, 2009; Shohat & Tamarkin, 1950) in which only the zeroth order moment is known (i.e., the support).

Third, by writing the dual of the above semi-infinite linear programming problem, we show that, if the nonlinearities in the measurement and process equations are polynomial and if the bounding sets for initial state, process and measurement noises are described by polynomial inequalities, then a feasible solution of the dual can be obtained by simply checking the non-negativity of a polynomial on a compact set described by polynomial inequalities. An approximation of this semi-infinite linear programming problem can be obtained by reformulating it as semidefinite programming by using the theory of *sum-of-squares* (SOS) polynomial optimization. We prove that the approximate solution is robust, in the sense that the computed half-space $\mathcal{H}$ is guaranteed to include $\mathcal{M}$, and so the membership set $\mathcal{X}$.

Fourth, we provide a procedure to determine the minimum-volume polytope $\mathcal{S}$ bounding $\mathcal{M}$. This procedure is based on a refinement of the algorithm originally proposed in Cerone, Piga, and Regruto (2012b) to compute an approximation of the minimum-volume polytope containing a given semialgebraic set. In particular, we use a Monte Carlo integration approach to compute an approximation of the volume of a polytope, and a greedy procedure to determine an outer-bounding polytope $\mathcal{S}$ as the intersection of a pre-specified number of half-spaces $\mathcal{H}_j$, where each half-space $\mathcal{H}_j$ is added to the description of $\mathcal{S}$ so to minimize the volume of the polytope including $\mathcal{M}$. This allows us to solve the set-membership estimation problem for polynomial non-linear systems very efficiently and through convex optimization.