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Distributed averaging with linear objective maps ☆

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ABSTRACT

A distributed averaging system is a linear multi-agent system in which agents communicate to reach an agreement (or a consensus) state, defined as the average of the initial states of the agents. Consider a more generalized situation in which each agent is given a nonnegative weight and the agreement state is defined as the weighted average of the initial conditions. We characterize in this paper the weighted averages that can be evaluated in a decentralized way by agents communicating over a directed graph. Specifically, we introduce a linear function, called the objective map, that defines the desired final state as a function of the initial states of the agents. We then provide a complete answer to the question of whether there is a decentralized consensus dynamics over a given digraph which converges to the final state specified by an objective map. In particular, we characterize not only the set of objective maps that are feasible for a given digraph, but also the consensus dynamics that implements the objective map. In addition, we present a decentralized algorithm to design the consensus dynamics.

1. Introduction

Distributed averaging has been recognized as an important step in a variety of decentralized and distributed algorithms, such as the rendezvous problem, distributed convex optimization or distributed sensing. We pose and solve in this paper the *weighted averaging* problem over a directed graph. Specifically, given a set of nonnegative weights assigned to the agents, we say that the agents reach a weighted consensus if they converge to the weighted average of their initial conditions—a formal definition to be given shortly. As is commonly done, we assume that the information flow in the system is described by a directed graph. Our goal is to determine which weighted averages can be computed for a given information flow. Furthermore, we describe how the agents communicate over the graph to *design* the dynamical system whose evolution reaches the desired agreement state. Computing a weighted average rather than a uniform average is a natural one when the agents in the system are not all on equal footing. For example, think of a rendezvous problem where the rendezvous position depends on the initial positions of only a small group of agents; of distributed sensing, where the weighting can be proportional to the accuracy of the sensing device, or of opinion dynamics, where participants may have different levels of influence on the decision process. Because of their broad relevance, a fair amount is already known about averaging and consensus algorithms. Indeed, questions concerning sufficient and/or necessary conditions for agents to reach consensus (Cao, Morse, & Anderson, 2008; Hendrickx & Tsitsiklis, 2011; Jadbabaie, Lin, & Morse, 2003; Liu, Nedić, & Başar, 2014; Moreau, 2005; Olfati-Saber & Murray, 2004; Ren & Beard, 2005; Tsitsiklis, 1984), questions concerning time delay (Cao et al., 2008; Olfati-Saber & Murray, 2004), consensus with quantized measurements (Başar, Etesami, & Olshevsky, 2014; Etesami & Başar, 2016; Kashyap, Başar, & Srikant, 2007), consensus with time-varying network topologies (Cao et al., 2008; Hendrickx & Tsitsiklis, 2011; Jadbabaie et al., 2003; Liu et al., 2014; Moreau, 2005; Olfati-Saber & Murray, 2004; Qu, Li, & Lewis, 2014; Ren & Beard, 2005; Tsitsiklis, 1984), and questions about estimating and/or improving convergence rate (Başar et al., 2014; De Gennaro & Jadbabaie, 2006; Etesami & Başar, 2016; Kim & Mesbahi, 2006; Liu, Mou, Morse, Anderson, & Yu, 2011; Qu et al., 2014) have all been treated to some degree.

Broadly speaking, the problem we address in this paper is one of *feasibility of an objective under decentralization constraints*. Similar questions, but involving system controllability (Chen, Belabbas, & Başar, 2015; Lin, 1974), stability of linear systems (Belabbas, 2013)

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and formation control (Chen, Belabbas, & Başar, 2015; Lorenzen & Belabbas, 2014) have also been investigated. While the general problem of feasibility of an objective under decentralization constraints is far from being completely understood, we shall see that a fairly complete characterization can be obtained for the present case, in both discrete- and continuous-time dynamics. However, still open questions remain, such as: How to handle negative weights? How to handle time-varying information flow graphs? How to make sure that no-agent can "game the system" and increase or decrease its assigned weight?

We next describe the model precisely. We assume that there are n agents $x_1, \ldots, x_n$ evolving in $\mathbb{R}^d$, and that the underlying network topology is specified by a directed graph (or simply *digraph*) $G = (V, E)$, with $V = \{1, \ldots, n\}$ the set of vertices and E the set of edges. We let V_i^- be a subset of V comprised of the outgoing neighbors of vertex i, i.e.,

$$V_i^- := \{j \in V \mid i \to j \in E\}$$

and we assume in this paper that each agent x_i can only observe its outgoing neighbors. The equations of motion for the n agents $x_1, \ldots, x_n$ are then given by

$$\frac{d}{dt} x_i = \sum_{j \in V_i^-} a_{ij}(x_j - x_i), \quad \forall i = 1, \ldots, n \tag{1}$$

with each a_{ij} a non-negative real number, which we call the interaction weight.

The objective of the system is characterized by a vector $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ of nonnegative entries. We define the *linear objective function* $f_w : \mathbb{R}^{n \times d} \longrightarrow \mathbb{R}^d$ as follows:

$$f_w(x_1, \ldots, x_n) := \sum_{i=1}^{n} w_i x_i.$$

The feasibility question we ask is the following: given a digraph $G = (V, E)$, and a weight vector $w = (w_1, \ldots, w_n)$ in $\mathbb{R}^n$, does there exist a set of non-negative interaction weights $\{a_{ij} \mid i \to j \in E\}$ such that for any initial condition $x_1(0), \ldots, x_n(0)$ in $\mathbb{R}^d$, all agents will converge to the same point in $\mathbb{R}^d$ specified by the objective map, i.e.,

$$\lim_{t \to \infty} x_i(t) = f_w(x_1(0), \ldots, x_n(0))$$

for all $i = 1, \ldots, n$? In other words, we require that all the agents not only reach an agreement, but also converge to a specific point which is a weighted sum of the initial positions of the agents. In the following section, we will convert this problem to one of asking whether there exists a sparse, infinitesimally stochastic matrix A with a fixed zero pattern (specified by the digraph) such that A has a simple zero eigenvalue with w the corresponding left eigenvector.

In the paper, we will provide a complete answer to the question of weighted averaging within model (1). In particular, we characterize both the set of objective maps which are feasible by choosing appropriate interaction weights and, reciprocally, the set of interaction weights for a feasible objective map. The results presented here expand on our earlier work (Chen, Belabbas, & Başar, 2015) by providing an analysis of the discrete-time case, a finer analysis of the set of interaction weights realizing a feasible linear objective map, a decentralized algorithm for implementing a particular choice of such set, and proofs that were left out. The remainder of the paper is organized as follows. In Section 2, we introduce definitions and establish results about the feasibility of linear objective maps over a given network topology. In Section 3, we characterize the sets of interaction weights associated with a feasible linear objective map. A decentralized algorithm for implementing a particular set of such interaction weights is then proposed in Section 4. We also illustrate the algorithm with simulations. We provide conclusions in the last section. The paper ends with Appendix which contains proofs of some technical results.

2. Feasibility of linear objective maps

In this section, we introduce the main definitions used in this work, formulate the weighted averaging problem in precise terms, and characterize the feasible linear objective maps over a given network topology.

2.1. Background and notations

We consider in this paper only *simple* directed graphs, that is directed graphs with no self-loops, and with at most one edge between each ordered pair of vertices. We denote by $G = (V, E)$ a directed graph where V is the vertex set and E is the edge set. Denote by $i \to j$ an edge of G, with i and j the start- and end-vertex of the edge, respectively. A vertex r is said to be a **root** of G if for all $i \in V$, there is a path from i to r. We say that G is **rooted** if it contains a root. Graphs with only one vertex are by convention rooted. We denote by $V_r \subset V$ the set of roots of G. The digraph G is **strongly connected** if, for any ordered pair of vertices (i, j), there is a path from i to j. In this case, all vertices of G are roots. It is well known that if the digraph G associated with system (1) is rooted, then all agents converge to the same state for all initial conditions. Conversely, if for any initial condition, all agents of system (1) converge to the same state, then the underlying digraph must be rooted (see, for example, Cao et al., 2008 and Ren & Beard, 2005). Hence, for the remainder of the paper, we consider only rooted digraphs as the underlying digraphs of system (1).

For a subset $V' \subset V$, we call G' a subgraph of G **induced by** V' if $G' = (V', E')$ and E' contains any edge of E whose start-vertex and end-vertex are in V'. We have the following definition:

Definition 1 (*Relevant Subset*)**.** Let $G = (V, E)$ be a rooted digraph. A subset V' of V is **relevant to** G (or simply **relevant**) if it satisfies the two conditions:

(a) The set V' is contained in the root set V_r;
(b) The subgraph G' induced by V' is strongly connected.

For G a digraph with n vertices, we can always let $V = \{1, 2, \ldots, n\}$. We denote by $\mathrm{Sp}[V]$ the unit simplex contained in $\mathbb{R}^n$ with vertices the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. For a subset $V' \subset V$, we define similarly $\mathrm{Sp}[V']$ as the convex hull of $\{e_i \mid i \in V'\}$:

$$\mathrm{Sp}[V'] := \left\{ \sum_{i \in V'} \alpha_i e_i \mid \alpha_i \geq 0, \sum_{i \in V'} \alpha_i = 1 \right\}.$$

We use the notation $\mathrm{Sp}(V')$ to denote the *interior* of $\mathrm{Sp}[V']$, which is defined by the same expression as above, but with $\alpha_i > 0$ for all $i \in V'$. If V' is comprised of only one vertex, say vertex i, we then set $\mathrm{Sp}[V'] = \mathrm{Sp}(V') = \{e_i\}$. We note here that if V_1 and V_2 are two different subsets of V, then $\mathrm{Sp}(V_1)$ and $\mathrm{Sp}(V_2)$ are disjoint. We introduce a similar notation to denote a convex cone. Let C_i, for $i = 1, \ldots, l$, be vectors in $\mathbb{R}^m$; we denote the convex cone spanned by C_i by

$$\mathrm{Co}[C_1, \ldots, C_l] := \left\{ \sum_{i=1}^{l} \alpha_i C_i \mid \alpha_i \geq 0 \right\}.$$

We denote its interior by $\mathrm{Co}(C_1, \ldots, C_l)$, which is defined by the same expression as above, but with $\alpha_i > 0$ for all $i = 1, \ldots, l$.

We further need some definitions on infinitesimally stochastic matrices (ISMs). First recall that a square matrix A is said to be an **infinitesimally stochastic matrix** if its off-diagonal entries are non-negative, and each of its rows sums to zero. We further need the following: