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Brief paper

Contraction and incremental stability of switched Carathéodory systems using multiple norms ☆

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ABSTRACT

In this paper, incremental exponential asymptotic stability of a class of switched Carathéodory nonlinear systems is studied based on the novel concept of measure of switched matrices via multiple norms and the transaction coefficients between these norms. This model is rather general and includes the case of staircase switching signals as a special case. Sufficient conditions are derived for incremental stability allowing for the system to be incrementally exponentially asymptotically stable even if some of its modes are unstable in some time periods. Numerical examples on switched linear systems with periodic switching and on the synchronization of switched networks of nonlinear systems are used to illustrate the theoretical results.

1. Introduction

Studying incremental stability of nonlinear systems is particularly important in many application areas, including observer design and, more recently, consensus and synchronization problems in network control where convergence analysis is a fundamental step (Russo & di Bernardo, 2009a,b; Russo, di Bernardo, & Slotine, 2011; Russo, di Bernardo, & Sontag, 2010, 2013; Wang & Slotine, 2005).

Since the early work by Lewis (1949), contraction theory has been highlighted as a promising approach to study incremental exponential asymptotic stability (δEAS) of nonlinear systems (Angeli, 2002; Forni & Sepulchre, 2014; Lohmiller & Slotine, 1998); see Jouffroy (2005) for an historical overview. In particular, as shown by Lohmiller and Slotine (1998), sufficient conditions for δEAS of a given nonlinear system over an invariant set of interest can be obtained by studying the matrix measure of its Jacobian induced by some vector norm. It is possible to prove, as done by Lohmiller and Slotine (1998) and Russo et al. (2010), that if such measure is negative definite in that set for all time then any two trajectories will exponentially converge towards each other; the rate of convergence being estimated by the negative upper bound on the Jacobian measure.

Numerous applications of contraction analysis have been presented in the literature from observer design to the synthesis of network control systems. See e.g. Forni and Sepulchre (2014), Lohmiller and Slotine (1998) and Russo et al. (2011). Remarkably, the problem of studying incremental stability of switched and hybrid systems has attracted relatively little attention despite the large number of potential applications, e.g. power electronic networks, variable structure systems, walking and hopping robots, to name just a few (Cortes, 2008; di Bernardo, Budd, Champneys, & Kowalczyk, 2008; Liberzon, 2003).

It has been suggested by di Bernardo, Liuzza, and Russo (2014) and Russo and di Bernardo (2011) that extending contraction analysis to this class of systems can be a viable and effective approach to obtain conditions for their incremental asymptotic stability. Related approaches include the work on convergence (Demidovich, 1967) of piecewise affine continuous systems presented by Pavlov and van de Wouw (2008), Pavlov, Pogromsky, van de Wouw, and Nijmeier (2007); Pavlov, Pogromsky, von de Wouw, Nijmeijer, and Rooda (2005a); Pavlov, van de Wouw, and

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Nijmeier (2005b) and the recent conference papers (di Bernardo & Fiore, 2014; di Bernardo & Liuzza, 2013).

One limitation of the existing extensions of contraction theory to switched systems, (e.g. di Bernardo et al., 2014), is that they rely on the use of a unique matrix measure to assess the Jacobian of each of the system modes. This is a particularly restricting assumption as it would be desirable to use measures induced by different norms to evaluate the Jacobian of each of the modes. This would correspond to studying incremental stability of the switched system with multiple incremental Lyapunov functions rather than using a common one (which is much harder to find).

The aim of this paper is to address this problem and present conditions for contraction and incremental stability of a large class of switched Carathéodory systems. The key idea is to define a novel concept of matrix measure via multiple norms and exploit the transaction coefficients between these norms. In so doing, sufficient conditions are derived for incremental stability that allow for a system to be δEAS even if some of its modes are unstable (or not contracting) over some time intervals. The theoretical results are illustrated via their applications to some representative examples, including synchronization in switched networks.

2. Preliminaries

We focus on switched dynamical systems of the form

$$\dot{x} = f(x, r(t)) \tag{1}$$

where $x \in \mathbb{R}^n$ and the switching signal $r(t)$ is assumed to be a real-valued piecewise continuous (PWC for short) function with respect to time: there exist countable discontinuous points $t_0 < t_1 < \cdots < t_i < \cdots$ such that $r(t_i\pm)$ exist and $r(t_i) = r(t_i+)$ for all t_i. A typical example is the staircase function $r(t) = \xi_i$, $\xi \in \mathbb{R}$, for $t_i \le t < t_{i+1}$, $i = 0, 1, \ldots$, for the increasing time sequence $\{t_j\}_{j\ge0}$, which has been widely used as switching signal in control systems (Liberzon, 2003).

Here we make the following hypothesis:

$\mathcal{H}_1$: For some $C \subset \mathbb{R}^n$, the vector field $f(x, r(t)) : C \times \mathbb{R}_{\ge0} \to \mathbb{R}^n$ is (i) continuous with respect to (x, r); (ii) continuously differentiable with respect to x, and (iii) there exists a Lebesgue measurable function $m(t)$ such that $|f(x, r(t))| \le m(t)$ for all $x \in C$ and $t \in \mathbb{R}_{\ge0}$.

It can be seen that under hypothesis $\mathcal{H}_1$, and with $r(t)$ defined as above, the vector field $f(x, r(t))$ defines a Carathéodory switched system (Filippov, 1988). It can be proved that, given an initial condition in C, a solution of a Carathéodory system exists and is unique (Hale, 1954). We define $\phi(t; t_0, x_0, r_t)$ as the solution of (1) with $x(t_0) = x_0$ and the switching signal $r(t)$, where r_t denotes the trajectory of $r(t)$ up to t, i.e., $r_t = \{r(s)\}_{t_0\le s\le t}$.

In this paper, $|\cdot|_\chi$ stands for a specific vector norm in Euclidean space and the matrix norm induced by it, which can be defined in different ways that are all equivalent. The *transaction coefficients* from the norm $|\cdot|_a$ to $|\cdot|_b$ are defined as $\beta_{ab} = \sup_{|x|_a=1} |x|_b$ (Bourbaki, 1978).

A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to *class* $\mathcal{K}$ if (I) it is strictly increasing; (II) $\alpha(0) = 0$. And, a continuous function $\beta(\rho, t) : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to *class* $\mathcal{KL}$ if (1) for each fixed t, the function $\beta(\rho, t)$ belongs to class $\mathcal{K}$; (2) for each fixed ρ, the function $\beta(\rho, t)$ is decreasing with respect to t and $\lim_{t\to\infty} \beta(\rho, t) = 0$. In addition, if a function $\beta(\rho, t)$ of class $\mathcal{KL}$ converges to 0 exponentially as $t \to \infty$, $\beta(\rho, t)$ is said to be of class $\mathcal{EKL}$. Here, we give the following definition of incremental stability from Angeli (2002) with modifications.

Definition 1. System (1) is said to be incrementally asymptotically stable (*δAS* for short) with $r(t)$ in the region $C \subset \mathbb{R}^n$ if there exists a function $\beta(s, t)$ of class $\mathcal{KL}$ such that for any initial data $x_0, y_0 \in C$ and starting time t_0, the following property holds

$$|\phi(t + t_0; t_0, x_0, r_t) - \phi(t + t_0; t_0, y_0, r_t)| \le \beta(|x_0 - y_0|, t)$$

for some norm $|\cdot|$. If $\beta(s, t)$ is picked independently of the initial time t_0, then system (1) is said to be incrementally uniformly asymptotically stable (*δUAS* for short). If $\beta(s, t)$ is of class $\mathcal{EKL}$, then system (1) is said to be incrementally exponentially asymptotically stable (*δEAS* for short) and incrementally uniformly exponentially asymptotically stable (*δUEAS* for short) if $\beta(s, t)$ is chosen independently of t_0.

Definition 2. A set $C \subset \mathbb{R}^n$ is said to be a κ-reachable set if there exists a continuously differentiable curve $\gamma(s) : [0, 1] \to C$ that links x_0 and y_0, i.e., $\gamma(0) = x_0$ and $\gamma(1) = y_0$, and satisfies $|\gamma'(s)|_{\chi(t_0)} \le \kappa|x_0 - y_0|_{\chi(t_0)}$, for all $s \in [0, 1]$ and some constant $\kappa > 0$, independently of the points x_0 and y_0 in C.

3. Switched matrix measures and general contraction analysis

The matrix measure induced by the vector norm $|\cdot|_\chi$, where χ is the index for the norm being used, is defined as

$$\mu_\chi(A) = \lim_{h\to0+} \frac{1}{h}\left[|(I_n + hA)|_\chi - 1\right]$$

for a square matrix $A \in \mathbb{R}^{n,n}$ and was used for the contraction analysis of smooth nonlinear systems, see e.g. Lohmiller and Slotine (1998).

Given a PWC function $\chi(t)$, the left (right) limit of $|\cdot|_{\chi(t)}$ at time t is defined as $\lim_{h\to0-} |x|_{\chi(t+h)}$ ($\lim_{h\to0+} |x|_{\chi(t+h)}$) if it exists for all $x \in R^n$, and is denoted by $|\cdot|_{\chi(t\pm)}$ respectively. We say that the switched norm $|\cdot|_{\chi(t)}$ is *continuous* at time t if $|x|_{\chi(t-)} = |x|_{\chi(t+)} = |x|_{\chi(t)}$ for all $x \in \mathbb{R}^n$, i.e. if $|\cdot|_{\chi(t)}$ is *left and right continuous*. We say that $|\cdot|_{\chi(t)}$ is *uniformly equivalent* if there exists a constant $D > 0$ such that $|x|_{\chi(t)} \le D|x|_{\chi(s)}$ for all $x \in \mathbb{R}^n$ and $t, s \in \mathbb{R}$.

We can now extend the definition of matrix measure to the case of multiple norms, taking the time-varying nature of $\chi(t)$ into consideration, as follows.

Definition 3. The switched matrix measure with respect to multiple norms $|\cdot|_{\chi(t)}$ is defined as

$$\nu_{\chi(t)}(A) = \overline{\lim_{h\to0+}} \frac{1}{h} \sup_{|x|_{\chi(t)}=1} \left[|(I_n + hA)x|_{\chi(t+h)} - 1\right] \tag{2}$$

if the limit exists, where $\overline{\lim}$ stands for the limit superior.

It can be seen that if $\chi(t)$ is constant over an interval, say $[t, t+\delta)$, then $\nu_{\chi(t)}(A) = \mu_{\chi(t)}(A)$ over that interval.

The existence of the switched matrix measure is related to the *partial differential* of the switched norm $|\cdot|_{\chi(t)}$ defined as follows

$$\overline{\partial}_t(|\cdot|_{\chi(t)}) = \overline{\lim_{h\to0+}} \sup_{|x|_{\chi(t)}=1} \frac{|x|_{\chi(t+h)} - 1}{h}.$$

We say that the multiple norm $|\cdot|_{\chi(t)}$ is *right regular* at time t if $\overline{\partial}_t(|\cdot|_{\chi(t)})$ exists. Thus, we have

Proposition 1. *If* $|\cdot|_{r(t)}$ *is right regular at* t*, then* (i) $|\cdot|_{\chi(t)}$ *is right continuous at* t*;* (ii) $\nu_{\chi(t)}(A)$ *exists at* t*.*

Proof. The first statement is straightforward from the definitions of $\overline{\partial}_t(|\cdot|_{\chi(t)})$ and right continuity of $|\cdot|_{\chi(t)}$.