Automatica 70 (2016) 66–73

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

On singular control problems with state constraints and regime-switching: A viscosity solution approach☆

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ARTICLE INFO

Article history:
Received 1 October 2014
Received in revised form
30 November 2015
Accepted 29 February 2016
Available online 13 April 2016

Keywords:
Constrained viscosity solution
Regime-switching diffusion
Singular stochastic control
Weak dynamic programming principle
Quasi-variational inequality

ABSTRACT

This paper investigates a singular stochastic control problem for a multi-dimensional regime-switching diffusion process confined in an unbounded domain. The objective is to maximize the total expected discounted rewards from exerting the singular control. Such a formulation stems from application areas such as optimal harvesting multiple species and optimal dividends payments schemes in random environments. With the aid of weak dynamic programming principle and an exponential transformation, we characterize the value function to be the unique constrained viscosity solution of a certain system of coupled nonlinear quasi-variational inequalities. Several examples are analyzed in detail to demonstrate the main results.

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1. Introduction

This paper is concerned with a class of singular stochastic control problems with state constraints. The controlled regime-switching diffusion process X and the singular control process Z take values in a convex cone $S \subset \mathbb{R}^n$. The control problem has the state process

$$X(t) = x + \int_0^t b(X(s), \alpha(s))ds + \int_0^t \sigma(X(s), \alpha(s))dW(s) - Z(t), \tag{1.1}$$

where $x \in \mathbb{R}^n$,, W is a d-dimensional standard Brownian motion, $\alpha(\cdot) \in \mathcal{M} = \{1, \ldots, m\}$ is a continuous-time Markov chain that is independent of the Brownian motion W and is generated by $Q = (q_{ij}) \in \mathbb{R}^{m\times m}$:

$$\mathbb{P}\{\alpha(t+\Delta t) = j | \alpha(t) = i, \alpha(s), s \le t\} = \begin{cases} q_{ij}\Delta t + o(\Delta t), & \text{if } j \neq i \\ 1 + q_{ii}\Delta t + o(\Delta t), & \text{if } j = i, \end{cases} \tag{1.2}$$

where $q_{ij} \ge 0$ for $i, j = 1, \ldots, m$ with $j \neq i$ and $q_{ii} = -\sum_{j\neq i} q_{ij} < 0$ for each $i = 1, \ldots, m$.

The control process $Z = (Z_1, \ldots, Z_n)'$ is further required to be an n-dimensional adapted, nondecreasing, and càdlàg stochastic process, which belongs to a control space $\mathcal{A}_{x,\alpha}$ to be defined later. The objective is to maximize the total discounted reward

$$J(x, \alpha, Z) := \mathbb{E}\left[\int_0^\infty e^{-rs} f(X^{x,\alpha}(s-), \alpha(s-)) \cdot dZ(s)\right], \tag{1.3}$$

where $r > 0$, $f : S \times \mathcal{M} \mapsto \mathbb{R}^n$ with f_i representing the state- and regime-dependent instantaneous marginal yields accrued from exerting the singular control $Z_i(t)$.

Such singular control problems (in various different settings) have been extensively studied in the literature. A partial list includes the monotone follower problems (Karatzas & Shreve, 1984), optimal harvesting problems (Alvarez & Shepp, 1998; Song, Stockbridge, & Zhu, 2011), portfolio selection management with transaction costs (Øksendal & Sulem, 2002), optimal partially reversible investment problem (Guo & Pham, 2005), etc. See also Haussmann and Suo (1995a; 1995b) for a general singular stochastic control problem for a multidimensional Itô diffusion

☆ The research of Qingshuo Song was supported in part by the RGC of CityU 109613 and the CityU SRG 7004241. The research of Chao Zhu was supported in part by the National Science Foundation under DMS-1108782 and a grant from the UWM Research Growth Initiative. The material in this paper was not presented before. This paper was recommended for publication in revised form by Associate Editor Valery Ugrinovskii under the direction of Editor Ian R. Petersen.

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http://dx.doi.org/10.1016/j.automatica.2016.03.017
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on a fixed time horizon, in which the existence of the optimal control and the characterization of the value function as the unique viscosity solution of a Hamilton–Jacobi–Bellman equation are established. Singular control problems with state constraints have drawn considerable interests in recent years; see, for example, Atar and Budhiraja (2006), Atar, Budhiraja, and Williams (2007) and Zariphopoulou (1992), among others.

Note that most, if not all, of the aforementioned literature on singular stochastic controls deal with Itô (jump) diffusions. One exception is our recent work Song et al. (2011), which studies an optimal harvesting problem of a single species living in random environments. Due to their capability of modeling complex systems with uncertainty, regime-switching models have drawn considerable attention from both researchers and practitioners in recent decades in a wide range of applications. Some of such examples can be found in mathematical finance (Zhang, 2001), ecosystem modeling (Zhu & Yin, 2009), stochastic manufacturing systems (Sethi & Zhang, 1994), risk management (Elliott & Siu, 2010; Zhu, 2011), to name just a few. In these systems, both continuous dynamics and discrete events coexist. In particular, the systems often display qualitative structural changes. Regime-switching models turn out to be quite versatile in capturing these inherent randomness. We refer to Mao and Yuan (2006) and Yin and Zhu (2010) for in-depth investigations of regime-switching diffusions.

This paper aims to investigate the singular control problem (1.3) in the setting of multi-dimensional regime-switching diffusion with state constraints. In particular, motivated by several simple yet nontrivial examples, we derive a strong comparison result from which we establish the existence and uniqueness of the constrained viscosity solution to (2.2).

Compared with the classical work on viscosity solution such as Crandall, Ishii, and Lions (1992) and others, the novelty and contribution of this work can be summarized as follows. In lieu of a single differential equation studied in the literature, this work deals with a coupled system of nonlinear second-order differential equations with gradient constraints. This is due to the presence of random environments or regime switching. This feature at one hand makes our model more appealing in real-world applications since it can naturally capture the qualitative structural changes of the systems; on the other hand, it adds much difficulty in the analysis. In particular, the function F defined in (2.3) is not proper in the sense of the User's Guide Crandall et al. (1992). Note that the properness was an essential assumption in the proof of the strong comparison result in Crandall et al. (1992). Here we need to carefully handle the coupling effect; see the proof of Theorem 11 for more details. Another noteworthy feature of this work is that we introduce an exponential transformation which allows us to handle both the gradient constraints as well as the polynomial growth condition on an unbounded domain for the solution of the quasi-variational inequalities (QVIs) (2.2).

The paper Ishii (1989) also contains comparison results (Theorems 7.1 and 7.3) for viscosity solutions of fully nonlinear second order PDEs $F(x, u, Du, D^2u) = 0$ in unbounded domain Ω. In deriving those results, in addition to the growth conditions of the solutions and some other conditions, the function $F(x, r, p, X)$ is assumed to be nondecreasing the r variable. As we mentioned earlier, our function F defined in (2.3) does not satisfy this assumption. Consequently we cannot apply the techniques in Ishii (1989) directly. Moreover, we need to take care of the gradient constraints in (2.2).

In addition, there are a few key differences between this paper and our previous work Song et al. (2011). By exploiting the advantage of dealing with a one-dimensional problem, Song et al. (2011) first derived a sufficient condition under which the value function is continuous, which, in turn, led to the existence proof of showing that it is *a* viscosity solution of the corresponding QVIs. In this paper, we use the weak dynamic programming principle to show directly that the value function is a constrained viscosity solution to the corresponding coupled system of QVIs (2.2). Moreover, with the aid of an exponential transformation, we establish a strong comparison theorem and hence the uniqueness part of constrained viscosity solution to (2.2). In this way, we obtain a complete characterization of the value function (2.1), i.e., it is the unique constrained viscosity solution to (2.2).

The rest of the paper is arranged as follows. Section 2 presents the precise formulation of the problem, followed by some preliminary results in Section 3. We recall the notion of constrained viscosity solution in Section 4, followed by several examples for illustration. Further, in Section 4, we establish the existence by showing that the value function V defined in (2.1) is a constrained viscosity solution of (2.2). The strong comparison result is arranged in Section 5. The paper is concluded with conclusions and remarks in Section 6.

To facilitate later presentation, we introduce some notations that will be used often in later sections. For a càdlàg (right continuous with left limits) function $\xi : [0, \infty) \mapsto \mathbb{R}^n$, we write $\Delta\xi(t) = \xi(t) - \xi(t-)$ for $t > 0$. As a convention, we set $\Delta\xi(0) = \xi(0)$. Throughout the paper, we use $x'y$ or $x \cdot y$ interchangeably to denote the inner product of vectors x and y. For any vectors $x, y \in \mathbb{R}^n$, $x \le y$ means $x_i \le y_i$ for every $i = 1, \ldots, n$. The space of $n \times n$ symmetric matrices is denoted by $\mathcal{S}_n$ and the family of positive definite symmetric matrices is denoted by $\mathcal{S}_n^+$. If $\phi : \mathbb{R}^n \to \mathbb{R}$ is sufficiently smooth, then $D_{x_i}\phi = \frac{\partial\phi}{\partial x_i}$, $D_{x_ix_j}\phi = \frac{\partial^2\phi}{\partial x_i \partial x_j}$, and $D\phi = (D_{x_1}\phi, \ldots, D_{x_n}\phi)'$ is the gradient of ϕ while $D^2\phi = (D_{x_ix_j}\phi)$ denotes the Hessian of ϕ. For any real-valued function f, we use f_* and f^* to denote the lower- and upper-semicontinuous envelopes of f, respectively. That is, $f_*(x) := \liminf_{y\to x} f(y)$ and $f^*(x) := \limsup_{y\to x} f(y)$. If $f : \mathbb{R}^n \times \mathcal{M} \mapsto \mathbb{R}$, with a slight abuse of notation, we define $f_*(x, i) := \liminf_{y\to x} f(y, i)$ and $f^*(x, i) := \limsup_{y\to x} f(y, i)$ for all $(x, i) \in \mathbb{R} \times \mathcal{M}$.

If B is a set, we use B^o and I_B to denote the interior and indicator function of B, respectively. Throughout the paper, we adopt the conventions that $\sup\emptyset = -\infty$ and $\inf\emptyset = +\infty$.

2. Formulation

Let ζ be a regime-switching diffusion process given by (1.1) with $Z \equiv 0$. Throughout the paper, we assume that the coefficients b and σ and the generator Q are such that for any initial condition $(x, \alpha) \in \mathbb{R}^n \times \mathcal{M}$, the solution $\zeta^{x,\alpha}$ to (1.1) (with $Z \equiv 0$) exists and is pathwise unique. Sufficient condition for existence and uniqueness for stochastic differential equations with regime switching can be found in, for example, Mao and Yuan (2006) and Yin and Zhu (2010).

Let $\mathcal{A}_{x,\alpha}$ denote the collection of all *admissible controls* with initial conditions (x, α), where $Z \in \mathcal{A}_{x,\alpha}$ satisfies

(i) for each $i = 1, \ldots, n$, $Z_i(t)$ is nonnegative, càdlàg and nondecreasing with respect to t,
(ii) $X(t) \in \bar{S}$ for all $t \ge 0$, and
(iii) $Z(t)$ is adapted to $\mathfrak{F}_t := \sigma\{W(s), \alpha(s), 0 \le s \le t\}$, where $\mathfrak{F}_0$ contains all $\mathbb{P}$-null sets. Moreover,

$$\mathbb{E}\left[\int_0^\infty e^{-rs} d\,|Z|\,(s)\right] < \infty.$$

Note that the state constraint is specified in condition (ii) above. Such a constraint and the consideration of the nonnegative and nondecreasing control processes Z_i, $i = 1, \ldots, n$ are motivated by many applications such as mathematical finance (Øksendal & Sulem, 2002), optimal harvesting problems (Alvarez & Shepp, 1998; Song et al., 2011), among others. We also assume the income

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