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Brief paper

# Delay-independent stability analysis of linear time-delay systems based on frequency discretization[☆]

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ABSTRACT

This paper studies strong delay-independent stability of linear time-invariant systems. It is known that delay-independent stability of time-delay systems is equivalent to some *frequency-dependent* linear matrix inequalities. To reduce or eliminate conservatism of stability criteria, the frequency domain is discretized into several sub-intervals, and *piecewise constant* Lyapunov matrices are employed to analyze the frequency-dependent stability condition. Applying the generalized Kalman–Yakubovich–Popov lemma, new necessary and sufficient criteria are then obtained for strong delay-independent stability of systems with a single delay. The effectiveness of the proposed method is illustrated by a numerical example.



## 1. Introduction

In many practical systems such as industrial processes and networked control systems, time-delay phenomena are inevitably encountered, and are often the key factor that affects the performance (Gu, Kharitonov, & Chen, 2003; Wang, Gao, & Qiu, 2015). Time-delay systems, although with a long history, are one of the most active topics in control and system theory in the past two decades, see Gu and Niculescu (2003) and Sipahi, Niculescu, Abdallah, Michiels, and Gu (2011) and the references therein. Even the most basic problem, stability analysis, of time-delay systems is still challenging due to its infinite-dimensional nature (Gu et al., 2003), and such study is still evolving (Sipahi et al., 2011). Sometimes, stability of systems can be maintained for all positive delays, thus giving the notion of *delay-independent* stability. This is in contrast to delay-dependent stability, in which case the system is stable for only certain range of delay values. In this paper, we focus on delay-independent stability.

The term "delay-independent stability" was introduced in Hale (1977), and many criteria have been developed for testing delay-independent stability of time-delay systems since then (see Delice & Sipahi, 2012, Souza, de Oliveira, & Palhares, 2009 for examples of more recent developments). Delay-independent stability itself includes two different notions, viz., *strong* delay-independent stability and *weak* delay-independent stability (see Definitions 1 and 2 in Section 2.1, respectively). The strong delay-independent stability, albeit being as a special case of the weak delay-independent one, is sufficiently general from a practical robustness point of view (Bliman, 2002). Necessary and sufficient criteria of delay-independent stability (both strong and weak) are often developed using a frequency domain method based on the characteristic equation. Some typical tools used include polynomial theory (Kamen, 1982), matrix pencil (Niculescu, 1998b), and robust control theory (Chen & Latchman, 1995). In addition to direct stability test, the necessary and sufficient conditions may also be useful in developing other simpler sufficient conditions that are easier to test, and uncovering their inherent conservatism.

A number of sufficient conditions for delay-independent stability can also be found in the literature (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994; Chen, Du, & Shafai, 1995; Kolmanovskii, Niculescu, & Richard, 1999). Although efforts in stability analysis are made mainly to derive necessary and sufficient conditions, the interest in some sufficient conditions is due to two factors. *First*, some sufficient conditions usually require much less computation than typical necessary and sufficient ones. *Second*, many sufficient conditions, especially those based on the Lyapunov stability theory

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(Boyd et al., 1994; Kolmanovskii et al., 1999), are easily adapted to other more complicated problems of time-delay systems. In fact, fruitful synthesis results on time-delay systems, whether delay-independent (Boyd et al., 1994; Shi, Boukas, & Agarwal, 1999; Wang, Huang, & Unbehauen, 1999; Wu & Grigoriadis, 2001) or delay-dependent (Du, Lam, & Shu, 2010; Fridman & Shaked, 2002; Gao & Li, 2011; He, Wu, She, & Liu, 2004; Li & Gao, 2011; Lin, Wang, & Lee, 2006; Palhares, Campos, Ekel, Leles, & D'Angelo, 2005), can be regarded as applications or extensions of simple linear matrix inequality (LMI) conditions (Agathoklis & Foda, 1989; Boyd et al., 1994).

In the paper, we will revisit the problem of strong delay-independent stability analysis of linear time-invariant systems with a state delay. Our attention will be focused on applying a *frequency-discretization* idea to develop new stability criteria in terms of linear matrix inequality (LMI). The advantage of the proposed stability criteria lies in the fact that they give a series of new sufficient conditions for systems with a single delay and become *nonconservative* as the frequency-discretization number goes to infinity, thus potentially less conservative than some typical sufficient LMI conditions in the literature. Numerical results will be provided to illustrate the improvement of the proposed method.

*Notation*: The superscripts "−1", "T", "∗" and "⊥" stand for inverse, transpose, conjugate transpose and null space of a matrix, respectively. $\mathbb{R}^{m\times n}$ ($\mathbb{C}^{m\times n}$) is the set of $m \times n$ real (complex) matrices. $\mathbb{C}_+$ denotes the closed right half plane of the complex plane, and $\mathbb{D}$ and $\partial\mathbb{D}$ denote the closed unit disc and the unit circle on the complex plane, respectively. The notation $P > 0$ ($\geq 0$) means that matrix $P$ is Hermitian positive definite (semi-definite). $\mathbf{S}_n$ and $\mathbf{H}_n$ are the sets of $n \times n$ symmetric and Hermitian matrices, respectively. $\mathbf{I}$ denotes an identity matrix with appropriate dimension. For a square matrix $A$, sym $\{A\}$ represents $(A^* + A)/2$. For a square matrix $A$, $\alpha(A)$ and $\rho(A)$ are the spectral abscissa and spectral radius of $A$, respectively, $\lambda(A)$ and $\sigma(A)$ are the eigenvalues and singular values of $A$, respectively. Matrix dimensions are assumed to be compatible for algebraic operations.

## 2. Main results

In this section, we present new stability conditions for systems with a single delay. Section 2.1 formulates the problem and provides some preliminaries. Section 2.2 comments some existing results for motivation. Technical details of the frequency-discretization idea and stability conditions are presented in Section 2.3, and numerical implementation of the stability conditions is discussed in Section 2.4.

### 2.1. Problem statement and preliminaries

Consider a linear continuous time-invariant system with a single delay described by the following delay-differential equation:

$$\dot{x}(t) = A_0x(t) + A_1x(t-d), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A_0$ and $A_1 \in \mathbb{R}^{n\times n}$ are known constant matrices, and $d \geq 0$ is the delay. Define a bivariate polynomial $c(s, z)$ as

$$c(s, z) \triangleq \det(s\mathbf{I} - A_0 - zA_1).$$

For a given delay $d$, it is known (Hale, 1977) that the asymptotic stability of system (1) is equivalent to

$$c(s, z) \neq 0, \quad \forall s \in \mathbb{C}_+ \quad \text{and} \quad z = \mathrm{e}^{-ds}. \tag{2}$$

In this paper, we are interested in system (1) whose stability is maintained for arbitrary delay $d \geq 0$. Two related notions of *delay-independent stability* for system (1) are defined as follows.

**Definition 1.** System (1) is said to be (weakly) delay-independently stable if the condition in (2) is satisfied for all $d \geq 0$.

**Definition 2.** System (1) is said to be strongly delay-independently stable if

$$c(s, z) \neq 0, \quad \forall (s, z) \in \mathbb{C}_+ \times \mathbb{D}. \tag{3}$$

According to the definition, strong delay-independent stability is defined by regarding $s$ and $z$ as independent of each other. As emphasized in Chen and Latchman (1995), strong delay-independent stability is stricter than the weak version in terms of the requirement at $s = 0$, where $z = 1$ in $c(s, z)$ can no longer be regarded as a variable independent of $s$ any more. However, the property of weakly delay-independent stability is not robust against perturbations of parameters $A_0$ and $A_1$ (Bliman, 2002). In this paper, we mainly consider strong delay-independent stability (but see Remark 3).

It is difficult to test strong delay-independent stability of system (1) directly according to its definition, because $c(s, z)$ is a bivariate polynomial. Define

$$S(s) \triangleq (s\mathbf{I} - A_0)^{-1}A_1, \qquad Z(z) \triangleq A_0 + zA_1.$$

The condition in (3) can be simplified to overcome this difficulty.

**Lemma 1.** *System (1) is strongly delay-independently stable if and only if either one of the following two equivalent conditions holds.*

(i)

$$\rho(S(s)) < 1, \quad \forall \Re(s) = 0 \tag{4}$$

*and*

$$\alpha(A_0) < 0. \tag{5}$$

(ii) $\alpha(Z(z)) < 0$ *for all* $z \in \partial\mathbb{D}$.

Condition (i) has been established in Agathoklis and Foda (1989), and Chen and Latchman (1995); and condition (ii) can be found in Agathoklis and Foda (1989), and Kamen (1982). In this paper, they will be used to develop novel and tractable stability criteria for system (1).

### 2.2. Observation and motivation

It has been well understood (Agathoklis & Foda, 1989; Boyd et al., 1994) that condition (i) of Lemma 1 holds if the following LMI holds for some $P_0 > 0$ and $P_1 > 0$:

$$\begin{bmatrix} A_0^{\mathrm{T}}P_0 + P_0A_0 + P_1 & P_0A_1 \\ A_1^{\mathrm{T}}P_0 & -P_1 \end{bmatrix} < 0, \tag{6}$$

which is known as two-dimensional (2-D) Lyapunov inequality (Agathoklis & Foda, 1989). This condition can be interpreted from two different points of view. *First*, according to the continuous-time bounded real lemma (Anderson & Vongpanitlerd, 1973), LMI (6) holds if and only if

$$\max_{\Re(s)=0} \sigma_{\max}(R_1S(s)R_1^{-1}) < 1; \qquad R_1^{\mathrm{T}}R_1 = P_1. \tag{7}$$

In view of the relationships:

$$\begin{aligned} \max_{\Re(s)=0} \rho(S(s)) &= \max_{\Re(s)=0} \rho\left(R_1S(s)R_1^{-1}\right) \\ &\leq \max_{\Re(s)=0} \sigma_{\max}(R_1S(s)R_1^{-1}), \end{aligned} \tag{8}$$

it can be seen that (7), or equivalently (6), is stricter than condition (i) in Lemma 1. This frequency-domain interpretation can be found, e.g., in Agathoklis and Foda (1989), Boyd et al. (1994) and Chen et al. (1995). *Second*, (6) can also be established from a time-domain point of view by using a *simple* Lyapunov–Krasovskii functional (Boyd et al., 1994). Both interpretations endow the condition in