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An improved method of ultimate bound computation for linear switched systems with bounded disturbances☆

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ABSTRACT

In this note, an improved method of ultimate bound computation for a linear switched system under arbitrary switching is presented. An ultimate bound for a linear switched system can be computed by solving a class of linear matrix inequalities. The effectiveness of the obtained results is illustrated by numerical examples.

1. Introduction

The stability of switched systems has attracted considerable research attention in recent years (Blanchini, Casagrande, & Miani, 2010; Decarlo, Branicky, Pettersson, & Lennartson, 2000; Dehghan & Ong, 2012; Liberzon & Morse, 1999; Lin & Antsaklis, 2009; Shorten, Wirth, Mason, Wulff, & King, 2007). In some cases, the asymptotic stability of switched systems cannot be ensured in the presence of disturbances. Hence, it is useful to study a practical stability problem for switched systems, such as an ultimate bound for the state trajectories. Some sufficient conditions for practical stability are provided in Haimovich and Seron (2010, 2013), Kofman, Haimovich, and Seron (2007) and Kofman, Seron, and Haimovich (2008), which rely on the existence of a transformation matrix that takes all matrices of the switched linear system into a form satisfying certain properties. In the literature (Blanchini et al., 2010; Dehghan & Ong, 2012), invariant sets for switched systems under dwell time switching are studied. These results can be applied to arbitrary switching systems by letting the dwell time be nearly zero. In this note, switched systems under arbitrary switching will be considered, which are given by

$$\dot{\boldsymbol{x}}(t) = A_{\sigma(t)}\boldsymbol{x}(t) + B_{\sigma(t)}\omega(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $\omega(t) \in \mathbb{R}^p$ is an external disturbance vector,

$$\sigma : \mathbb{R}_+ \to \{1, 2, \dots, s\} \tag{2}$$

is the switching function, $A_r \in \mathbb{R}^{n\times n}$, $B_r \in \mathbb{R}^{n\times p}$, $r \in [1, s]$ are known constant matrices. The disturbance $\omega(t) = [\omega_1(t) \;\cdots \omega_p(t)]^{\mathrm{T}}$ is unknown but assumed to be bounded by a given constant vector $\bar{\omega} = [\bar{\omega}_1 \;\cdots \bar{\omega}_p]^{\mathrm{T}} \in \mathbb{R}^p$ with $\bar{\omega}_i > 0$, $i \in [1, p]$, i.e., for all $t \geq 0$,

$$|\omega_i(t)| \leq \bar{\omega}_i, \quad \forall i \in [1, p]. \tag{3}$$

2. Preliminaries

In the following, for real symmetric matrices X, Y, the notations $X \geq 0$ and $Y > 0$ mean that the matrix X is positive semi-definite and the matrix Y is positive definite, respectively. For any vectors $\boldsymbol{x} = [x_1 \;\cdots\; x_n]^{\mathrm{T}} \in \mathbb{R}^n$ and $\boldsymbol{y} = [y_1 \;\cdots\; y_n]^{\mathrm{T}} \in \mathbb{R}^n$, the notation $\boldsymbol{x} \geq \boldsymbol{y}$ means that $x_i - y_i \geq 0$ for all $i \in [1, n]$. Given $a \in \mathbb{C}$, $\mathbb{R}\mathrm{e}(a)$ denotes the real part of a. Let 0_n denote the n-dimensional vector, whose elements are all 0. Let $0_{n\times n}$ denote a $n \times n$ dimensional matrix, whose elements are all 0. Given a vector $\boldsymbol{x} = [x_1 \cdots x_n]^{\mathrm{T}} \in \mathbb{C}^n$, let $\boldsymbol{x}^{\mathrm{H}}$ denote the conjugate transpose of the vector $\boldsymbol{x}$, let $\tilde{\boldsymbol{x}} = |\boldsymbol{x}|$ denote a vector $\tilde{\boldsymbol{x}} = [\tilde{x}_1 \cdots \tilde{x}_n]^{\mathrm{T}} \in \mathbb{R}^n$,

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whose elements are defined as $\tilde{x}_i = |x_i|, i \in [1, n]$. Given a matrix $U = [u_{ij}] \in \mathbb{C}^{n\times n}$, let U^{H} denote the conjugate transpose of the matrix U, let $\tilde{U} = |U|$ denote a matrix $\tilde{U} = [\tilde{u}_{ij}] \in \mathbb{R}^{n\times n}$, whose elements are defined as $\tilde{u}_{ij} = |u_{ij}|, i, j \in [1, n]$. Given matrices $U_1 = [u_{1ij}], \ldots, U_s = [u_{sij}] \in \mathbb{R}^{n\times n}$, let $U = \max_{r\in[1,s]}\{U_r\}$ denote a matrix $U = [u_{ij}] \in \mathbb{R}^{n\times n}$, whose elements $u_{ij}, i, j \in [1, n]$ are defined as $u_{ij} = \max_{r\in[1,s]}\{u_{rij}\}, i, j \in [1, n]$. Given vectors $\boldsymbol{x}_1 \cdots \boldsymbol{x}_s \in \mathbb{R}^n$, $\boldsymbol{x}_r = [x_{r1} \cdots x_{rn}]^{\mathrm{T}}, r \in [1, s]$, let $\boldsymbol{x} = \max_{r\in[1,s]}\{\boldsymbol{x}_r\}$ denote a vector $\boldsymbol{x} = [x_1 \cdots x_n]^{\mathrm{T}} \in \mathbb{R}^n$, whose elements $x_i, i \in [1, n]$ are defined as $x_i = \max_{r\in[1,s]}\{x_{ri}\}, i \in [1, n]$. Given a matrix $U = [u_{ij}] \in \mathbb{C}^{n\times n}$, let $\bar{U} = \mathcal{M}(U)$ denote a matrix $\bar{U} = [\bar{u}_{ij}] \in \mathbb{R}^{n\times n}$, whose elements are defined as follows:

$$\bar{u}_{ij} = \begin{cases} |u_{ij}|, & \text{if } i \neq j, \\ \mathbb{R}\mathrm{e}(u_{ij}), & \text{if } i = j. \end{cases}$$

Lemma 1 (Haimovich and Seron, 2010). *Given a system (1)–(3) and an invertible matrix $T = (M + \mathbf{j}N) \in \mathbb{C}^{n\times n}$, $M, N \in \mathbb{R}^{n\times n}$, if the matrix Λ is Hurwitz, then it follows that*

$$\lim_{t\to\infty} \sup |T^{-1}\boldsymbol{x}(t)| \leq \zeta, \tag{4}$$

where $\boldsymbol{z}_r = \max_{\omega:|\omega|\leq\bar{\omega}} |T^{-1}B_r\omega|$, $\boldsymbol{z} = \max_{r\in[1,s]}\{\boldsymbol{z}_r\}$, $\Lambda = \max_{r\in[1,s]}\{\Lambda_r\}$, $\zeta = [\zeta_1 \cdots \zeta_n]^{\mathrm{T}} \in \mathbb{R}^n$, $\zeta = -\Lambda^{-1}\boldsymbol{z}$ and $\Lambda_r = [\lambda_{rij}] \in \mathbb{R}^{n\times n}$ is defined as $\Lambda_r = \mathcal{M}(T^{-1}A_rT), r \in [1, s]$.

The algorithms to calculate a transform matrix T are proposed in the paper Haimovich and Seron (2010, 2013). Then, the ultimate bound for system (1)–(3) can be calculated, i.e.,

$$\lim_{t\to\infty} \sup |\boldsymbol{x}(t)| \leq [\bar{x}_1 \cdots \bar{x}_n]^{\mathrm{T}}, \tag{5}$$

where $\bar{\boldsymbol{x}} = [\bar{x}_1 \cdots \bar{x}_n]^{\mathrm{T}} \in \mathbb{R}^n$, $\bar{\boldsymbol{x}} = |T|\zeta$, ζ is defined as in Lemma 1. It can been seen that (4) is equivalent to

$$\lim_{t\to\infty} \sup |e_i^{\mathrm{T}}T^{-1}\boldsymbol{x}(t)| \leq \zeta_i \quad \forall i \in [1, n], \tag{6}$$

where $e_i \in \mathbb{R}^n$ denotes a n-dimensional vector, whose ith element is 1 and others are 0, $\zeta_i, i \in [1, n]$ are defined as in Lemma 1. Without loss of generality, in the following, we assume that $\zeta_i > 0$ and $\bar{x}_i > 0, i \in [1, n]$.

The main purpose of this note is to obtain a small ultimate bound for the system (1)–(3) based on the results of Lemma 1.

Lemma 2. *Suppose that the state trajectories of system (1)–(3) satisfy (4), where $T \in \mathbb{C}^{n\times n}$ and $\zeta \in \mathbb{R}^n$ are defined as in Lemma 1. For the given matrices $Q_r \in \mathbb{R}^{n\times n}$ with $Q_r = Q_r^{\mathrm{T}}$, $\tilde{Q}_r \in \mathbb{R}^{n\times n}$, $\Xi_r \in \mathbb{R}^{n\times n}$, $\tilde{\Xi}_r \in \mathbb{R}^{n\times n}$, $G_r = [g_{rij}] \in \mathbb{R}^{n\times n}$, $\tilde{G}_r = [\tilde{g}_{rij}] \in \mathbb{R}^{n\times n}$, $r \in [1, s]$, if*

$$g_{rij} \geq |f_{rij}| \quad \forall i, j \in [1, n],\ r \in [1, s], \tag{7}$$

$$\tilde{g}_{rij} \geq |\tilde{f}_{rij}| \quad \forall i, j \in [1, n],\ r \in [1, s] \tag{8}$$

then it follows that

$$\lim_{t\to\infty} \sup |\boldsymbol{x}^{\mathrm{T}}(t)Q_r\boldsymbol{x}(t)| \leq \zeta^{\mathrm{T}}G_r\zeta \quad \forall r \in [1, s], \tag{9}$$

$$\lim_{t\to\infty} \sup |\boldsymbol{x}^{\mathrm{T}}(t)\tilde{Q}_rB_r\omega(t)| \leq \zeta^{\mathrm{T}}\tilde{G}_r\boldsymbol{z}_r \quad \forall r \in [1, s], \tag{10}$$

where $F_r = [f_{rij}] \in \mathbb{C}^{n\times n}$, $\tilde{F}_r = [\tilde{f}_{rij}] \in \mathbb{C}^{n\times n}$, $F_r = T^{\mathrm{H}}(Q_r + \mathbf{j}\Xi_r)T$, $\tilde{F}_r = T^{\mathrm{H}}(\tilde{Q}_r + \mathbf{j}\tilde{\Xi}_r)T$, $r \in [1, s]$. Also $\boldsymbol{z}_r$, $r \in [1, s]$ are defined as in Lemma 1.

Proof. It follows from $\boldsymbol{x}(t) \in \mathbb{R}^n$ and $\omega(t) \in \mathbb{R}^p$ that for all $r \in [1, s]$,

$$|\boldsymbol{x}^{\mathrm{T}}(t)\tilde{Q}_rB_r\omega(t)| \leq |\boldsymbol{x}^{\mathrm{T}}(t)(\tilde{Q}_r + \mathbf{j}\tilde{\Xi}_r)B_r\omega(t)|, \tag{11}$$

$$|\boldsymbol{x}^{\mathrm{T}}(t)\tilde{Q}_rB_r\omega(t)| \leq |\boldsymbol{x}^{\mathrm{T}}(t)(T^{-1})^{\mathrm{H}}| \cdot |\tilde{F}_r| \cdot |T^{-1}B_r\omega(t)|. \tag{12}$$

Then, using (8) and the definition of $\boldsymbol{z}_r$ in Lemma 1, we obtain for all $r \in [1, s]$,

$$|\boldsymbol{x}^{\mathrm{T}}(t)\tilde{Q}_rB_r\omega(t)| \leq |\boldsymbol{x}^{\mathrm{T}}(t)(T^{-1})^{\mathrm{H}}|\tilde{G}_r\boldsymbol{z}_r. \tag{13}$$

It follows from (4) that for all $r \in [1, s]$,

$$\lim_{t\to\infty} \sup |\boldsymbol{x}^{\mathrm{T}}(t)(T^{-1})^{\mathrm{H}}|\tilde{G}_r\boldsymbol{z}_r \leq \zeta^{\mathrm{T}}\tilde{G}_r\boldsymbol{z}_r. \tag{14}$$

Using (13) and (14), we can obtain (10). Similarly, we may obtain (9). This completes the proof. □

3. Main results

Theorem 1. *Suppose that the state trajectories of system (1)–(3) satisfy (4), where $T \in \mathbb{C}^{n\times n}$ and $\zeta \in \mathbb{R}^n$ are defined as in Lemma 1. For a given vector $\boldsymbol{P} = \boldsymbol{P}_R + \mathbf{j}\boldsymbol{P}_I \in \mathbb{C}^n$ with $\boldsymbol{P}_R, \boldsymbol{P}_I \in \mathbb{R}^n$, if there exist matrices $Q_r \in \mathbb{R}^{n\times n}$, $\tilde{Q}_r \in \mathbb{R}^{n\times n}$, $\Xi_r \in \mathbb{R}^{n\times n}$, $\tilde{\Xi}_r \in \mathbb{R}^{n\times n}$, $G_r = [g_{rij}] \in \mathbb{R}^{n\times n}$, $\tilde{G}_r = [\tilde{g}_{rij}] \in \mathbb{R}^{n\times n}$, $r \in [1, s]$, diagonal matrix $D_r \in \mathbb{R}^{p\times p}$ with $D_r \geq 0$, $r \in [1, s]$ and scalars $\alpha_r > 0$, $r \in [1, s]$, $\beta > 0$ such that (7), (8),*

$$\Pi_r \leq 0 \quad \forall r \in [1, s], \tag{15}$$

$$\Gamma_r < \beta \quad \forall r \in [1, s], \tag{16}$$

then, it follows that

$$\lim_{t\to\infty} \sup |\boldsymbol{P}^{\mathrm{H}}x(t)| \leq \beta, \tag{17}$$

where $F_r = [f_{rij}] \in \mathbb{C}^{n\times n}$, $\tilde{F}_r = [\tilde{f}_{rij}] \in \mathbb{C}^{n\times n}$, $r \in [1, s]$ are defined as in Lemma 2, $\tilde{P} = \boldsymbol{P}_R\boldsymbol{P}_R^{\mathrm{T}} + \boldsymbol{P}_I\boldsymbol{P}_I^{\mathrm{T}}$, $\Gamma_r = \bar{\omega}^{\mathrm{T}}D_r\bar{\omega} + \zeta^{\mathrm{T}}G_r\zeta + 2\zeta^{\mathrm{T}}\tilde{G}_r\boldsymbol{z}_r$,

$$\Pi_r = \begin{bmatrix} \alpha_r[A_r^{\mathrm{T}}\tilde{P} + \tilde{P}A_r] + \beta^{-1}\tilde{P} - Q_r & \alpha_r\tilde{P}B_r - \tilde{Q}_rB_r \\ \alpha_rB_r^{\mathrm{T}}\tilde{P} - B_r^{\mathrm{T}}\tilde{Q}_r^{\mathrm{T}} & -D_r \end{bmatrix}.$$

Proof. It follows from (16) that there exists a positive scalar ϵ such that

$$\Gamma_r + 2\epsilon < \beta \quad \forall r \in [1, s]. \tag{18}$$

Using (3) and the definition of D_r, we obtain that for all $t \geq 0$,

$$\omega^{\mathrm{T}}(t)D_r\omega(t) \leq \bar{\omega}^{\mathrm{T}}D_r\bar{\omega} \quad \forall r \in [1, s]. \tag{19}$$

Using (7), (8) and the results of Lemma 2, we obtain (9) and (10). It follows from (9), (10) and (19) that there exists a scalar $t_\epsilon < \infty$ such that for all $t \geq t_\epsilon$,

$$\delta_r(t) \leq \Gamma_r + \epsilon \quad \forall r \in [1, s], \tag{20}$$

where $\delta_r(t) = \omega^{\mathrm{T}}(t)D_r\omega(t) + \boldsymbol{x}^{\mathrm{T}}(t)Q_r\boldsymbol{x}(t) + 2\boldsymbol{x}^{\mathrm{T}}(t)\tilde{Q}_rB_r\omega(t)$.

We construct the function $V(\boldsymbol{x}(t)) = \boldsymbol{x}^{\mathrm{T}}(t)\tilde{P}\boldsymbol{x}(t)$. Then, for the system (1)–(3), it follows that

$$\dot{V}(\boldsymbol{x}(t)) = (\alpha_{\sigma(t)})^{-1} \times [\xi^{\mathrm{T}}(t)\Pi_{\sigma(t)}\xi(t) - \boldsymbol{x}^{\mathrm{T}}(t)\beta^{-1}\tilde{P}\boldsymbol{x}(t) + \delta_{\sigma(t)}(t)], \tag{21}$$

where $\xi(t) = [\boldsymbol{x}^{\mathrm{T}}(t)\ \omega^{\mathrm{T}}(t)]^{\mathrm{T}}$. It follows from (15) that $\xi^{\mathrm{T}}(t)\Pi_{\sigma(t)}\xi(t) \leq 0$. Then, using (20), (21) and $\alpha_r > 0$, $r \in [1, s]$, we can obtain

$$\dot{V}(\boldsymbol{x}(t)) \leq (\alpha_{\sigma(t)})^{-1}[\beta - \epsilon - \beta^{-1}V(\boldsymbol{x}(t))] \quad \forall t \geq t_\epsilon. \tag{22}$$

In the following, we will show that there exists a scalar t_β such that

$$V(x(t)) \leq \beta^2 \quad \forall t \geq t_\beta. \tag{23}$$

First, we will use the method of contradiction to prove that there exists a scalar t_β with $t_\epsilon \leq t_\beta < \infty$ such that $V(\boldsymbol{x}(t_\beta)) < \beta^2$.