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Technical communique

Finite-time sliding mode control synthesis under explicit output constraint☆

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ABSTRACT

This work deals with the input–output finite-time stabilization problem for a class of nonlinear systems by employing sliding mode control (SMC) approach. A suitable SMC law is designed to ensure that the state trajectories can be driven onto the specified sliding surface during the assigned finite-time interval. Moreover, some parameters-dependent sufficient conditions are derived such that the input–output finite-time stability (IO-FTS) during both *reaching phase* and *sliding motion phase* are attained. Simulation results are provided to illustrate the proposed approach.

1. Introduction

Input–output finite-time stability (IO-FTS) of a system concerns the *quantitative* behavior of the output variables over an assigned finite-time interval. That is, a system is said to be input–output finite-time stable if its output (weighted) norm does not exceed an assigned threshold β (i.e., an *explicit* output constraint condition) during the specified time interval $[0, T]$ (Amato, Ambrosino, & Cosentino, 2010; Amato, Carannante, & de Tommasi, 2012, 2014; Song, Niu, & Jia, 2015).

Sliding mode control (SMC), as an effective robust control strategy for systems subjected to parameter uncertainties and external disturbances, has attracted considerable attention, see Basin and Rodríguez-Ramírez (2011), Chen, Niu, and Zou (2013), Wu, Su, and Shi (2012), Xia, Lu, and Zhu (2013) and the references therein. However, it should be pointed out that, in almost all aforementioned works on SMC, the behavior of sliding mode dynamics was considered within a sufficiently long (in principle *infinite*) time interval and there was no any constraint on transient dynamics. Apparently, this case is not always true in some practical applications, such as when considering IO-FTS, in which the specified finite-time interval T and the explicit output constraint scalar β should be taken into account simultaneously. Specifically, the following two questions should be answered for the input–output finite-time stabilization via SMC:

(1) For any specified finite (possibly *short*) interval T, is it possible to design a SMC law such that the state trajectories can be driven onto the sliding surface in a finite-time T^* with $T^* \leq T$; if so, how to design?
(2) For an assigned threshold β, how to guarantee that the output (weighted) norm does not exceed β during both *reaching phase* and *sliding motion phase*?

In this technical communique, we focus on addressing the input–output finite-time stabilization problem for a class of uncertain nonlinear systems by using SMC approach. The above two questions will be answered in the following design (see Theorems 1 and 2, respectively).

Notation. $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of the corresponding matrix. $\|\cdot\|$ and $|\cdot|$ denote, respectively, the Euclidean norm and 1-norm of a vector (sum of absolute values) or its induced matrix norm. For a real matrix, A^{T} represents the transpose of A, and we denote $\mathrm{He}\{A\} = A + A^{\mathrm{T}}$. The shorthand "diag{·}" denotes a block diagonal matrix. In symmetric block matrices, the symbol "$\star$" is used as an ellipsis for terms induced for symmetry.

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2. System descriptions and definition

Consider the nonlinear system described by

$$\Sigma : \begin{cases} \dot{x}(t) = f(x(t), w(t)) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state; $u(t) \in \mathbb{R}^m$ is the input; $w(t) \in \mathbb{R}^r$ is the disturbance input. The unknown function $f(x(t), w(t))$ satisfies the conic sector constraint (ElBsat & Yaz, 2013):

$$\begin{aligned} &\left\| f(x(t), w(t)) - \left[\bar{A}(t)x(t) + Fw(t)\right] \right\| \\ &\quad \le \left\| \bar{A}_r(t)x(t) + F_r w(t) \right\|, \end{aligned} \tag{2}$$

where $\bar{A}(t) = A + A_\Delta(t)$ and $\bar{A}_r(t) = A_r + A_{r\Delta}(t)$. Matrices A, B, C, F, A_r, and F_r are assumed to be known. $A_\Delta(t)$ and $A_{r\Delta}(t)$ are parameter uncertainties satisfying the following norm-bounded condition:

$$\begin{bmatrix} A_\Delta(t) & A_{r\Delta}(t) \end{bmatrix} = \begin{bmatrix} M & M_r \end{bmatrix} E(t)N, \tag{3}$$

where M, M_r and N are known real matrices and $E(t)$ is unknown with $\|E(t)\| \le 1$.

By using the conic-type constraint condition (2), the system dynamic (1) can be rewritten as

$$\bar{\Sigma} : \begin{cases} \dot{x}(t) = \bar{A}(t)x(t) + Fw(t) + g(x(t), w(t)) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{4}$$

with $g(x(t), w(t)) = f(x(t), w(t)) - \left[\bar{A}(t)x(t) + Fw(t)\right]$. It is easily shown that $g(x(t), w(t))$ satisfies

$$\|g(x(t), w(t))\|^2 \le \|\bar{A}_r(t)x(t) + F_r w(t)\|^2. \tag{5}$$

Definition 1 (*IO-FTS* *Amato et al., 2010, 2012, 2014*)**.** Given a time interval $[t_0, t_1]$, an explicit output constraint scalar $\beta > 0$, disturbance signals $\mathbb{W}$ defined over $[t_0, t_1]$, a weighted matrix $R > 0$. System $\bar{\Sigma}$ with $u(t) = 0$ is said to be IO-FTS with respect to $(\beta, [t_0, t_1], R, \mathbb{W})$ if, under zero initial condition $x(t_0) = 0$, it holds that

$$w(t) \in \mathbb{W} \Rightarrow y^{\mathrm{T}}(t)Ry(t) \le \beta, \quad t \in [t_0, t_1]. \tag{6}$$

In this work, we consider the class of norm bounded square integrable signals over $[0, T]$, defined as follows:

$$\mathbb{W} \triangleq \left\{ w(\cdot) \in \mathcal{L}_{2,[0,T]} : \int_0^T w^{\mathrm{T}} w dt \le \delta \right\}, \tag{7}$$

with a known scalar $\delta > 0$.

3. Sliding surface design

In this work, our aim is to cope with the IO-FTS problem of SMC for nonlinear system $\bar{\Sigma}$, wherein the sliding function $s(t)$ is chosen as

$$s(t) = Lx(t) - \int_0^t L(A + B\mathcal{K})x(\tau)d\tau, \tag{8}$$

where the matrix $\mathcal{K}$ will be designed later (in Theorem 2), and the matrix L is chosen so that LB is nonsingular, which can be attained by choosing $L = B^{\mathrm{T}}\mathcal{X}$ with $\mathcal{X} > 0$, since B is assumed to be of full column rank.

4. Reachability with $T^* \le T$

In this section, a suitable sliding mode controller is designed to drive the state trajectories onto the specified sliding surface $s(t) = 0$ in a finite time T^* and then are maintained there over the rest time interval $[T^*, T]$.

Theorem 1. *Consider system (Σ) in (1)–(3). The reachability of the specified sliding surface (8) can be ensured in a finite time T^* with $T^* \le T$ by the SMC law:*

$$u(t) = \mathcal{K}x(t) - \eta(t)\mathrm{sgn}(s(t)), \tag{9}$$

where the robust term $\eta(t)$ is given by

$$\eta(t) = \varsigma + v\|w(t)\| + d\|x(t)\|, \tag{10}$$

in which $v = \|(B^{\mathrm{T}}\mathcal{X}B)^{-1}B^{\mathrm{T}}\mathcal{X}\|(\|F\| + \|F_r\|)$, $d = \|(B^{\mathrm{T}}\mathcal{X}B)^{-1} B^{\mathrm{T}}\mathcal{X}\|(\|M\|\,\|N\| + \|M_r\|\,\|N\| + \|A_r\|)$, and $\varsigma > 0$ is an adjustable parameter satisfying

$$\varsigma \ge \frac{\lambda_{\max}[(B^{\mathrm{T}}\mathcal{X}B)^{-1}]}{T}\|B^{\mathrm{T}}\mathcal{X}x(0)\|. \tag{11}$$

Proof. Choose the Lyapunov function as $U(s) = \frac{1}{2}s^{\mathrm{T}}(B^{\mathrm{T}}\mathcal{X}B)^{-1}s$, with $\mathcal{X} > 0$. By considering expressions (3), (5) and (8), we have

$$\begin{aligned} \dot{U}(s) &= s^{\mathrm{T}}(B^{\mathrm{T}}\mathcal{X}B)^{-1}[B^{\mathrm{T}}\mathcal{X}(A_\Delta - B\mathcal{K})x + B^{\mathrm{T}}\mathcal{X}Bu \\ &\quad + B^{\mathrm{T}}\mathcal{X}Fw + B^{\mathrm{T}}\mathcal{X}g(x, w)] \\ &\le \|s\|\,(v\|w\| + d\|x\|) - s^{\mathrm{T}}\mathcal{K}x + s^{\mathrm{T}}u. \end{aligned} \tag{12}$$

Substituting (9), (12) and noting $\|s\| \le |s|$, we have

$$\dot{U}(s) \le -\varsigma\|s\| \le -\frac{\varsigma}{\tilde{\gamma}}\sqrt{U(s)}, \tag{13}$$

where $\tilde{\gamma} = \sqrt{\frac{\lambda_{\max}\left[(B^{\mathrm{T}}\mathcal{X}B)^{-1}\right]}{2}} > 0$.

It is easily obtained from (13) that there exists a time $T^* \le \frac{2\tilde{\gamma}}{\varsigma}\sqrt{U(0)}$ such that $U(s) = 0$, and consequently $s(t) = 0$, for $t \ge T^*$.

Furthermore, by the fact that $U(0) \le \frac{\lambda_{\max}\left[(B^{\mathrm{T}}\mathcal{X}B)^{-1}\right]}{2}\|s(0)\|^2$, it yields

$$T^* \le \frac{\lambda_{\max}[(B^{\mathrm{T}}\mathcal{X}B)^{-1}]}{\varsigma}\|B^{\mathrm{T}}\mathcal{X}x(0)\|. \tag{14}$$

By considering (11) for ς, we have from (14) that $T^* \le T$. This means that, for any given finite-time T, the state trajectories can be driven onto the predefined sliding surface $s(t) = 0$ in a finite time T^* with $T^* \le T$. ■

Remark 1. It should be pointed out that the designed SMC law (9)–(10) depends on the assigned time T via the selection criterion (11) on the scalar ς. This strategy ensures that the states can be driven onto the specified sliding surface in the interval $[0, T^*]$ with $T^* \le T$. The selection on ς in this work is different from some existing works, e.g., Chen et al. (2013) and Wu et al. (2012), in which similar parameter was selected arbitrarily.

5. SMC synthesis with IO-FTS

5.1. *IO-FTS over reaching phase* $[0, T^*]$

In the sequel, it will be shown that the closed-loop system is IO-FTS during *reaching phase*, i.e., in the time interval $[0, T^*]$. By substituting SMC law (9) into (4), the closed-loop system over $[0, T^*]$ is obtained by:

$$\tilde{\Sigma}_{[0,T^*]} : \begin{cases} \dot{x} = \hat{A}x + Fw + g(x, w) - B\eta_s, \\ y = Cx, \end{cases} \tag{15}$$

where $\hat{A} = \bar{A} + B\mathcal{K}$ and $\eta_s \triangleq \eta \cdot \mathrm{sgn}(s)$.

Lemma 1. *Given a feasible scalar $\alpha > 0$. The resulting closed-loop system $\tilde{\Sigma}_{[0,T^*]}$ in (15) is IO-FTS with respect to $(\beta, [0, T^*], R, \mathbb{W})$, if there exist matrices $\mathcal{K} \in \mathbb{R}^{m\times n}$, $W > 0$ and $P > 0$, and scalars $\epsilon > 0$*