Automatica 65 (2016) 160–163

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

New results on pseudospectral methods for optimal control[☆]

Xiaojun Tang[1], Zhenbao Liu, Yu Hu

School of Aeronautics, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, China

ARTICLE INFO

Article history:
Received 22 February 2015
Received in revised form
18 August 2015
Accepted 2 November 2015

Keywords:
Optimal control
Pseudospectral methods
Equivalence

ABSTRACT

In this note, the equivalence between differential and integral pseudospectral methods is justified from the distinctive perspective of *Birkhoff interpolation* for collocation at the Jacobi–Gauss and flipped Jacobi–Gauss–Radau points. Furthermore, an exact, efficient, and stable approach is presented for computing the associated pseudospectral differentiation/integration matrices even at millions of points. These new results will contribute to the deeper understanding of pseudospectral methods and their practical applications in optimal control.

1. Introduction

In recent years, pseudospectral methods such as the Lobatto pseudospectral method (Elnagar, Kazemi, & Razzaghi, 1995; Elnagar & Razzaghi, 1997; Fahroo & Ross, 2001), the Gauss pseudospectral method (Benson, 2004; Benson, Huntington, Thorvaldsen, & Rao, 2006; Huntington, 2007), and the Radau pseudospectral method (Garg, 2011; Garg, Hager, & Rao, 2011; Garg et al., 2011), have been extensively used in the numerical solution of optimal control problems. Basically there are two primary implementation forms for pseudospectral methods: differential and integral. In a differential pseudospectral method, the differential constraints are directly collocated at a specified set of points via pseudospectral differentiation matrices (PDMs). In an integral pseudospectral method, the differential constraints are first recast into integral constraints which are then collocated at the specified set of points via pseudospectral integration matrices (PIMs). As a result, the accuracy of pseudospectral methods relies heavily on that of PDMs/PIMs. So far, to the extent of our knowledge, very little work has been done on the computation of PDMs/PIMs.

The above three pseudospectral methods employ the Legendre–Gauss–Lobatto (LGL), the Legendre–Gauss (LG), and the Legendre–Gauss–Radau (LGR) points, respectively. It is noted that the LGR points are defined on the half open interval $[-1, +1)$ or $(-1, +1]$, and thus, can be classified into standard LGR and flipped LGR (FLGR) points, respectively. Moreover, all of these points are associated with the Legendre polynomials which are a subclass of the more general Jacobi polynomials. Although differential and integral pseudospectral methods are quite different, recent work (Garg et al., 2010) has shown that they are equivalent for collocation at the LG and FLGR points. Therefore, a question arises naturally here: does such equivalence still hold for collocation at the Jacobi–Gauss (JG) and flipped Jacobi–Gauss–Radau (FJGR) points? Moreover, how does one compute the associated PDMs/PIMs with high accuracy?

In this note, we take a distinctive route to justify the above equivalence from the perspective of *Birkhoff interpolation* (see, e.g., Costabile & Longo, 2010, Lorentz, Jetter, & Riemenschneider, 2009, Shi, 2003, Wang, Samson, & Zhao, 2013, Wang, Samson, & Zhao, 2014, Wang, Zhao, & Zhang, 2014, Zhang, 2012), and present an exact, efficient, and stable approach for computing the PDMs/PIMs. The rest of this paper is organized as follows. The definitions and computation of PDMs/PIMs for the JG and FJGR points are presented in Section 2. In Section 3, the mentioned equivalence is proved using Birkhoff interpolation. Finally, Section 4 contains some concluding remarks.

2. Definitions and computation of PDMs/PIMs

In this section, the definitions and computation of PDMs/PIMs for the JG and FJGR points are presented, respectively.

☆ This work was partly supported by the Natural Science Basic Research Plan in Shaanxi Province of China (Program No. 2014JQ8366) and the Fundamental Research Foundation of Northwestern Polytechnical University (Grant No. JCY20130103). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Carlo Fischione under the direction of Editor André L. Tits.

E-mail addresses: nputxj@nwpu.edu.cn (X. Tang), liuzhenbao@nwpu.edu.cn (Z. Liu), huyu1974@nwpu.edu.cn (Y. Hu).

[1] Tel.: +86 13571893786; fax: +86 29 88492344.

http://dx.doi.org/10.1016/j.automatica.2015.11.035
0005-1098/

2.1. Definitions of PDMs/PIMs

Definition 1. The PDM/PIM for the JG points of $\{\tau_i \in (-1, +1)\}_{i=1}^N$ with $-1 = \tau_0 < \tau_1 < \cdots < \tau_{N+1} = +1$ are defined, respectively, as

$$\begin{aligned} &\boldsymbol{D}^\star_{ki} \triangleq \dot{\mathcal{L}}^\star_i(\tau_k), \\ &(k = 1, 2, \ldots, N, \quad i = 0, 1, \ldots, N) \end{aligned} \tag{1a}$$

$$\begin{aligned} &\boldsymbol{I}_{ki} \triangleq \int_{-1}^{\tau_k} \mathcal{L}_i(\tau)\,\mathrm{d}\tau, \\ &(k = 1, 2, \ldots, N+1, \quad i = 1, 2, \ldots, N) \end{aligned} \tag{1b}$$

where $\{\mathcal{L}^\star_i(\tau)\}_{i=0}^N$ are the Nth-order Lagrange interpolating polynomials associated with the interpolating points $\{\tau_i\}_{i=0}^N$, defined as

$$\mathcal{L}^\star_i(\tau) \triangleq \prod_{j=0, j\neq i}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad i = 0, 1, \ldots, N. \tag{2}$$

Similarly, $\{\mathcal{L}_i(\tau)\}_{i=1}^N$ are the $(N-1)$th-order Lagrange interpolating polynomials associated with the interpolating points $\{\tau_i\}_{i=1}^N$, defined as

$$\mathcal{L}_i(\tau) \triangleq \prod_{j=1, j\neq i}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad i = 1, 2, \ldots, N. \tag{3}$$

Definition 2. The PDM/PIM for the FJGR points of $\{\hat{\tau}_i \in (-1, +1]\}_{i=1}^N$ with $-1 = \hat{\tau}_0 < \hat{\tau}_1 < \cdots < \hat{\tau}_N = +1$ are defined, respectively, as

$$\begin{aligned} &\hat{\boldsymbol{D}}^\star_{ki} \triangleq \dot{\hat{\mathcal{L}}}^\star_i(\hat{\tau}_k), \\ &(k = 1, 2, \ldots, N, \quad i = 0, 1, \ldots, N) \end{aligned} \tag{4a}$$

$$\begin{aligned} &\hat{\boldsymbol{I}}_{ki} \triangleq \int_{-1}^{\hat{\tau}_k} \hat{\mathcal{L}}_i(\tau)\,\mathrm{d}\tau, \\ &(k, i = 1, 2, \ldots, N) \end{aligned} \tag{4b}$$

where $\{\hat{\mathcal{L}}^\star_i(\tau)\}_{i=0}^N$ and $\{\hat{\mathcal{L}}_i(\tau)\}_{i=1}^N$ are defined, respectively, in Eqs. (2) and (3) with the interpolating points being $\{\hat{\tau}_i\}_{i=0}^N$ and $\{\hat{\tau}_i\}_{i=1}^N$.

2.2. Computation of PDMs/PIMs

Now, we describe an exact, efficient, and stable approach for computing the PDMs/PIMs defined above. We start with the barycentric Lagrange interpolation and the associated barycentric weights for the Jacobi-type points, which are important pieces of the puzzle for our new approach.

It is well known that the Lagrange interpolating polynomials of Eq. (3) can be represented in the following barycentric form (Berrut & Trefethen, 2004)

$$\mathcal{L}_i(\tau) = \frac{\xi_i}{\tau - \tau_i} \Bigg/ \sum_{j=1}^{N} \frac{\xi_j}{\tau - \tau_j}, \quad i = 1, 2, \ldots, N \tag{5}$$

where $\{\xi_i\}_{i=1}^N$ are the barycentric weights, defined as (Berrut & Trefethen, 2004)

$$\xi_i \triangleq \frac{1}{\prod_{j=1, j\neq i}^{N} (\tau_i - \tau_j)}, \quad i = 1, 2, \ldots, N. \tag{6}$$

The barycentric Lagrange interpolating polynomials of Eq. (5) are scale-invariant, and thus, avoid any problems of underflow and overflow (Berrut & Trefethen, 2004). Furthermore, they are numerically stable for evaluating the polynomial interpolants at points $\tau \in [-1, +1]$ through any set of interpolating points with a small Lebesgue constant (Higham, 2004; Webb, Trefethen, & Gonnet, 2012). However, direct calculation of the barycentric weights using Eq. (6) suffers from significant numerical errors when the number of interpolating points is large. Fortunately, for the Jacobi-type interpolating points, we have the following lemma.

Lemma 3. *The barycentric weights for the JG, FJGR, and JGR points are given, respectively, as*

$$\xi_i = (-1)^{i-1}\sqrt{(1-\tau_i^2)\,\omega_i}, \quad i = 1, 2, \ldots, N \tag{7a}$$

$$\hat{\xi}_i = \begin{cases} (-1)^{i-1}\sqrt{(1+\hat{\tau}_i)\,\hat{\omega}_i}, & i = 1, 2, \ldots, N-1 \\ (-1)^{N-1}\sqrt{2(\alpha+1)\,\hat{\omega}_N}, & i = N \end{cases} \tag{7b}$$

$$\check{\xi}_i = \begin{cases} \sqrt{2(\beta+1)\,\check{\omega}_1}, & i = 1 \\ (-1)^{i-1}\sqrt{(1-\check{\tau}_i)\,\check{\omega}_i}, & i = 2, 3, \ldots, N \end{cases} \tag{7c}$$

where $\{\tau_i, \omega_i\}_{i=1}^N$, $\{\hat{\tau}_i, \hat{\omega}_i\}_{i=1}^N$, and $\{\check{\tau}_i, \check{\omega}_i\}_{i=1}^N$ are the sets of JG, FJGR, and JGR points and quadrature weights, respectively, associated with the Jacobi weight function $\omega^{(\alpha,\beta)}(\tau) = (1-\tau)^\alpha(1+\tau)^\beta$.

Proof. See Wang, Huybrechs, and Vandewalle (2014, Corollaries 2.3 and 3.7). □

Using the barycentric Lagrange interpolation, we can compute the PDM of Eq. (1a) as (Berrut & Trefethen, 2004)

$$\boldsymbol{D}^\star_{ki} = \begin{cases} \dfrac{\xi^\star_i/\xi^\star_k}{\tau_k - \tau_i}, & k \neq i \\ -\displaystyle\sum_{j=0, j\neq k}^{N} \boldsymbol{D}^\star_{kj}, & k = i \end{cases} \tag{8}$$

where $\{\xi^\star_i\}_{i=0}^N$ are the barycentric weights associated with the interpolating points $\{\tau_i\}_{i=0}^N$. However, this conventional approach needs to calculate $\{\xi^\star_i\}_{i=0}^N$ directly which is error-prone for large N as mentioned before. In this note, we present a very smart approach for computing the PDMs/PIMs and the main results are summarized in the following two theorems.

Theorem 4. *The PDM for the JG points can be computed exactly as*

$$\boldsymbol{D}^\star_{ki} = \begin{cases} -\displaystyle\sum_{j=1}^{N} \boldsymbol{D}^\star_{kj}, & i = 0 \quad \text{(a)} \\ \dfrac{\delta_{ki} + (\tau_k - \tau_0)\boldsymbol{D}_{ki}}{\tau_i - \tau_0}, & i \neq 0 \quad \text{(b)} \end{cases} \tag{9}$$

where

$$\begin{aligned} \boldsymbol{D}_{ki} &\triangleq \dot{\mathcal{L}}_i(\tau_k), \quad k, i = 1, 2, \ldots, N \\ &= \begin{cases} \dfrac{\xi_i/\xi_k}{\tau_k - \tau_i}, & k \neq i \\ -\displaystyle\sum_{j=1, j\neq k}^{N} \boldsymbol{D}_{kj}, & k = i \end{cases} \end{aligned} \tag{10}$$

where $\{\xi_i\}_{i=1}^N$ are calculated using Eq. (7a). The same result also holds for the FJGR points.

Proof. It follows from Eqs. (2) and (3) that

$$\mathcal{L}^\star_i(\tau) = \frac{\tau - \tau_0}{\tau_i - \tau_0} \cdot \mathcal{L}_i(\tau), \quad i = 1, 2, \ldots, N. \tag{11}$$