Automatica 67 (2016) 33–45

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Optimal switching strategy of a mean-reverting asset over multiple regimes☆

Kiyoshi Suzuki [1]

Portfolio Consulting Department, Nomura Securities Co. LTD., Urbannet Otemachi Building, 2-2-2, Otemachi, Chiyoda-ku, Tokyo, 100-8130, Japan

ARTICLE INFO

Article history:
Received 9 January 2014
Received in revised form
8 August 2015
Accepted 27 November 2015
Available online 5 February 2016

Keywords:
Optimal multiple switching problem
Hamilton–Jacobi–Bellman (HJB) variational inequality
Switching regions
Viscosity solutions
Ornstein–Uhlenbeck process
Hermite function
Pairs trading strategy

ABSTRACT

We solve optimal iterative three-regime switching problems with transaction costs, with investment in a mean-reverting asset that follows an Ornstein–Uhlenbeck process and find the explicit solutions. The investor can take either a long, short or square position and can switch positions during the period. Modeling the short sales position is necessary to study optimal trading strategies such as the pair trading. Few studies provide explicit solutions to problems with multiple (more than two) regimes (states). The value function is proved to be a unique viscosity solution of a Hamilton–Jacobi–Bellman variational inequality (HJB-VI). Multiple-regime switching problems are more difficult to solve than conventional two-regime switching problems, because they need to identify not only when to switch, but also where to switch. Therefore, multiple-regime switching problems need to identify the structure of the continuation/switching regions in the free boundary problem for each regime. If the number of the states N is two, only two regions have to be identified, but if $N = 3$, ${}_NP_2 = 6$ regions have to be detected. We identify the structure of the switching regions for each regime using the theories related to the viscosity solution approach.

1. Introduction

As a typical example of an investment strategy to a mean-reverting asset or portfolio, we study the pairs trading strategy, which is used by many hedge funds. Consider two similar stocks that is highly correlated. Assume that the spread between the two stock prices fluctuates randomly and the spread has a long-run mean. Sometimes the spread process diverges from the long-run mean and sometimes it converges. If the spread widens, the expensive stock is sold and the cheap stock is purchased. As the spread narrows again, profit is taken by unwinding the pairs position. This paper provides an analytical framework for such an investment strategy.

We solve a general finite iterative problem with n options to switch. In the literature, the optimal selling problem ($n = 1$) was studied first, followed by the optimal switching problem. For example, Johnson (2006) studies these cases step by step, which include the optimal selling problem with the initial long position, the optimal purchase problem with the initial square position, the problem to purchase with the option to sell afterwards, the problem to sell with the option to buy afterward and the infinite open–close switching problem for the same criterion, but they do not analyze general finite number iterative switching problems. In this paper, general iterative switching problems are consistently analyzed.

The optimal switching problem determines the optimal sequence of stopping times and regimes (or states, modes) to switch for a stochastic process. This is a classical and important problem, studied extensively since the late 1970s. It has recently received renewed and increasing interest because of its many applications in economics and finance, particularly real options. In typical problems of real-option, the states of entry ($\xi = 1$) and exit ($\xi = 0$) are available. Bayraktar and Egami (2010), Dixit (1989), Dixit and Pindyck (1994), Duckworth and Zervos (2001), Hamadene and Jeanblanc (2007), Metcalf and Hassett (1995), Pham (2009), Pham and Vath (2007), Sarkar (2003), Thompson (2002), Tsekrekos (2010) and Zervos (2003) study two regime cases. Indeed, most studies consider the two-regime case.

As our focus is on financial assets, especially pair spread process, we therefore study the regime-switching problem with three positions: square, long and short sales. The model is more flexible and more applicable to businesses than are the conventional

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Valery Ugrinovskii under the direction of Editor Ian R. Petersen.

E-mail address: abbeyroad20@outlook.com.

[1] Tel.: +81 3 6703 1597; fax: +81 3 6703 5814.

http://dx.doi.org/10.1016/j.automatica.2015.12.023
0005-1098/

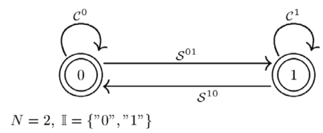

Fig. 1. Diagrams for the two-state case.

two-regime switching models. The paper by Pham, Vath, and Zhou (2009) studies the problem with three regimes, with some assumptions regarding the profit (reward) function that our profit function does not satisfy (different profit functions with identical diffusion regimes). The underlying state variable is restricted to the positive area in their model, and while they study abstract mathematical models, we solve the problem for practical applications. We believe that our model will be the first explicit solution to the three-regime switching problem in which investors can take short (reverse) positions (as well as the square positions) of a mean-reverting asset in a continuous-time market. The main contribution of this paper is that we solve a three-regime switching problem analytically, which is more complex than two-regime switching problems in the following sense. Normally, in the optimal switching problem, we need to detect a continuation region $\mathcal{C}^{\xi}$ for each state $\xi \in \mathbb{I}$ and a switching region $\mathcal{S}^{\xi\hat{\xi}}$ for each state $\xi, \hat{\xi} \in \mathbb{I}$, where the current state ξ switches to another state $\hat{\xi}$. If the current position z is in the region $\mathcal{C}^{\xi}$, the corresponding value function $v(z, \xi, n)$ with n switching options should be greater than any other value functions in the state $\hat{\xi} \neq \xi$ less than the transaction costs, $v(z, \hat{\xi}, n-1) - K, \hat{\xi} \in \mathbb{I}$. That is, for each $\xi \in \mathbb{I}$, $z \in \mathcal{C}^{\xi} \iff v(z, \xi, n) > \max_{\hat{\xi} \in \mathbb{I}}\{v(z, \hat{\xi}, n-1) - K\}$. Otherwise ($z \notin \mathcal{C}^{\xi}$), at least one of the comparisons should satisfy the equality, i.e., $\mathcal{S}_n^{\xi\hat{\xi}} = \{z | v(z, \xi, n) = v(z, \hat{\xi}, n-1) - K\}$ for $\xi, \hat{\xi} \in \mathbb{I}$ (see (27)). Therefore, generally, if the number of states in the universe $\mathbb{I}$ is N, for each state $\xi \in \mathbb{I}$, the domain $\mathbb{R}$ of the variable z is divided into one continuation region $\mathcal{C}^{\xi}$ and $N-1$ switching regions $\mathcal{S}^{\xi\hat{\xi}}$, i.e., $\mathbb{R} = \mathcal{C}^{\xi} \cup \left(\bigcup_{\hat{\xi} \neq \xi} \mathcal{S}_n^{\xi\hat{\xi}}\right)$. Therefore, in general, we need to identify $\mathcal{N} \equiv N +_N P_s - N = N(N-1)$ regions to solve the N-regime switching problem. The structure of the continuation/switching regions is illustrated in diagram form for $N = 2, 3$ in Fig. 1, Fig. 2. If $N = 2$, then $\mathcal{N} = 2$; therefore, identifying only two $\mathcal{C}^{\xi}$'s is enough to solve the problem. That is, the switching regions $\mathcal{S}^{\xi\hat{\xi}}$ ($\xi, \hat{\xi} \in \mathbb{I}$) are detected automatically ($\mathcal{S}^{\xi\hat{\xi}} = \mathbb{R} - \mathcal{C}^{\xi}$) in this case. This is a degenerate case. However, if $N = 3$, then $\mathcal{N} = 6$, and we need to find not only when to switch, but also where to switch at the same time. That is, some of $\mathcal{S}^{\xi\hat{\xi}}$ should be calculated directly, which was unnecessary for the case of $N = 2$. That is the most remarkable feature of the optimal switching problem when the universe $\mathbb{I}$ of the states is extended to $N \geq 3$, which is the difficulty with such a multiple-regime switching problem, in sharp contrast to the two-regime switching problem. Sometimes identifying the structure of the continuation/switching regions in the free boundary problem for each state is complicated. If $N \geq 4$, i.e., $\mathcal{N} \geq 12$, this problem is even more difficult. In this paper, we identify the structure of the switching regions for $N = 3$ using the theories related to the viscosity solution approach. Although the model of the problem itself is basic, the explicit solution needs several steps of nontrivial theorems and lemmas with such advanced techniques.

The efficient management of a power plant, for example, requires multiple production modes to include intermediate operating modes corresponding to different subsets of turbine running. Such an example of multiple switching problem applied to energy tolling agreements is considered by Deng and Xia (2005) and Ludkowski (2005), who focus mainly on a numerical resolution based on Monte Carlo regressions. Yet, few studies provide a complete analytical treatment and mathematical solution to the optimal multiple-regime switching problem.

Applications of the optimal switching problem to the financial asset management industry have been studied since the 2000s. In Eloe, Liu, Yatsuki, Yin, and Zhang (2008), Guo and Zhang (2005), Pemy and Zhang (2006) and Zhang (2001), the optimal selling rule for the geometric Brownian motion process is analyzed. Our model includes the optimal selling problem as a special case of $n = 1$; i.e., with only one option to switch.

Use of a stationary process for the price spread is usually termed co-integration. Some studies consider optimal investment problems of co-integrated assets in a continuous-time model. Among them Chiu and Wong (2011, 2012), Mudchanatongsuk, Primbs, and Wong (2008) and Tourin and Yan (2013) consider the problem in the context of the Merton-type stochastic control approach, based on infinitesimal rebalance without transaction costs, assuming a portfolio of co-integrated assets. On the other hand, Bayraktar and Egami (2010), Nguyen, Tie, and Zhang (2013), Song and Zhang (2013), Tsekrekos (2010) and Zhang and Zhang (2008) consider co-integrated assets in the context of an optimal switching problem with transaction costs, in which rebalancing occurs at discrete time intervals. The latter approach can capture the life cycle of a particular mean-reverting asset and is motivated by "when to buy and when to sell a particular position", while the former approach is motivated by the optimal portfolio weights of the constituents' pair positions. Our model is based on the latter approach. In terms of the approaches to the investment strategy of the pairs trading, the latter framework might be more natural and practical for investment managers than other stochastic control approaches if their intuition is based on the life-cycle of the pair positions and we believe that the resulting optimal switching strategy is what the asset management industry has long wanted to establish. That is, fund managers' typical intuition is that once they find the distortion in the market, they construct the position of a pair and wait until the distortion reverts to the normal state and then, the position is unwound with taking profit. That is considered to be the typical lifecycle of the strategy and the investment manager's major concern is when to construct the position and when to unwind it depending on the states of the market and the position.

In terms of the underlying mean-reverting assets, the paper by Bayraktar and Egami (2010) studies the optimal switching problem for the underlying Ornstein–Uhlenbeck process with two regimes, with an absorbing boundary at $X = 0$, where X is the asset price level, which should be positive. As our focus is on the pair spread process, our analysis naturally considers the three regimes including short sales (reverse position), and the feasible region of the underlying state variable X is whole $\mathbb{R}$, including the negative area. Nguyen et al. (2013), Song and Zhang (2013) and Zhang and Zhang (2008) also study the optimal entry–exit problem for a mean-reverting asset. However, they study only the two-regime case and the feasible domain of the process is restricted to the upper half-plane of $\mathbb{R}$. The remarkable feature of the pairs-trading strategy in practice is to be able to reverse the positions symmetrically, and also take the square position in the same strategy. Although in some of the above studies you can reverse the position if all the positions are multiplied by -1, they are still two-regime switching problems. Note that, in general, any optimal switching strategies cannot take the long, short and square positions in the same model, without having the structure with $N \geq 3$ and $\mathcal{N} \geq 6$ discussed above (Fig. 2; see the necessary condition (25)).

The classical dynamic programming approach is used only when it is assumed a priori that the value function is smooth enough. This is not necessarily true. The critical step in the classical approach to dynamic programming consists in proving that, given a smooth solution to the HJB equation, this candidate coincides with the value function. This result is called verification theorem,