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Moving-horizon estimation with guaranteed robustness for discrete-time linear systems and measurements subject to outliers ☆

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ABSTRACT

An approach to state estimation for discrete-time linear time-invariant systems with measurements that may be affected by outliers is presented by using only a batch of most recent inputs and outputs according to a moving-horizon strategy. The approach consists in minimizing a set of least-squares cost functions in which each measure possibly contaminated by outlier is left out in turn. The estimate that corresponds to the lowest cost is retained and propagated to the next time instant, where the procedure is repeated with the new information batch. The stability of the estimation error for the proposed moving-horizon estimator is proved under mild conditions concerning the observability of the free-noise state equation and the selection of a tuning parameter in the cost function. Robustness is guaranteed with sufficiently large outliers. The effectiveness of the proposed method as compared with the Kalman filter is shown by means of a numerical example.

1. Introduction

In numerous applications there exists the problem of dealing with large deviations in the measurements because of sensor malfunctions, wrong replacement of measures, or large non-Gaussian noises. These abnormal signals are usually called outliers in many different fields such as process control (Pearson, 2002), heart surgery (Ortmaier, Groger, Boehm, Falk, & Hirzinger, 2005), intrusion detection (Zhang, Zulkernine, & Haque, 2008), environmental monitoring (Garces & Sbarbaro, 2011), positioning (Fallahi, Cheng, & Fattouche, 2012), cloud management (Meng & Liu, 2013), and fault detection (Ferdowsi, Jagannathan, & Zawodniok, 2014). Various filtering methods have been proposed to attenuate or detect outliers (see, e.g., Gandhi & Mili, 2010 and the references therein). In this paper, a more general problem is addressed that consists in estimating the state variables of a linear system by means of measures possibly corrupted by outliers. The estimation is performed by using a moving-horizon approach, which will be set in such a way to make it robust to outliers.

The problem of estimating the state variable of a linear system with output contaminated by outliers can be treated by using the Kalman filter with some convenient adjustment. As is well-known, under the assumption that initial state and disturbances are white Gaussian stochastic processes, the best estimator in the sense of the minimization of the expected quadratic estimation error is the Kalman filter. Such an estimator is recursive in that the new output is processed by iterating the estimate update based on the current residual, i.e., the output error given by the difference between the measure and its prediction obtained from the last state estimate. Thus, one may check abnormal residuals via a threshold test to skip the Kalman estimate update with such residuals. This procedure can be motivated from a theoretical point of view by using the maximum likelihood criterion (Alessandri & Awawdeh, 2014).

The first ideas about what is currently denoted as moving-horizon estimation (MHE) are presented in Jazwinski (1968). MHE consists in performing state estimation by using a limited amount of most recent information. The state estimates are obtained by minimizing a least-squares cost function with a batch of the inputs and outputs according to a sliding-horizon strategy. Constraints on the state variables may be easily taken into account since the optimization is carried on line. The first results on MHE for linear systems (Alessandri, Baglietto, & Battistelli, 2003; Rao, Rawlings, & Lee, 2001) have been extended to nonlinear (see, e.g., Alessandri, Baglietto, & Battistelli, 2008; Alessandri, Baglietto, Battistelli, & Gaggero, 2011; Fagiano & Novara, 2013; Guo & Huang, 2013; Rao, Rawlings, & Mayne, 2003) and large-scale systems (Farina, Ferrari-Trecate, & Scattolini, 2010).

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Outliers are particular type of uncertainty that prevent an estimator from ensuring guaranteed performances (Matasov & Samokhvalov, 1996). Robustness is thus a fundamental requirement in the design of filters for uncertain systems. A method to enhance the robustness of the Kalman filter in the presence of outliers is presented in Gandhi and Mili (2010). In Akkaya and Tiku (2008) and Shi, Chen, and Shi (2013) statistical tests are proposed that are less sensitive to abnormal noises. For the same reasons, an l_1 loss function is more suitable for the purpose of identification with measures affected by outliers (Lauer, Bloch, & Vidal, 2011; Xu, Bai, & Cho, 2014). The reader is referred to Rousseeuw and Leroy (1987) for a complete review of the most important methods of regression that account for robustness to outliers.

Based on the preliminary results by Alessandri and Awawdeh (2014), here we focus on MHE for linear discrete-time systems with measurements contaminated by outliers. Toward this end, first we will prove the stability of the estimation error and, second, the robustness to outliers. Conditions for the stability of moving-horizon estimators for uncertain linear systems are reported in Alessandri, Baglietto, and Battistelli (2012), where explicit bounding sequences are provided thanks to the adoption of worst-case cost functions. Unfortunately, such cost functions are not helpful in case of measurements affected by outliers, thus a different criterion is proposed here. More specifically, at each time instant we separately minimize a set of least-squares cost functions, where the measurements that can be affected by outliers are left out in turn. Then, we choose the minimizer associated with the lowest cost, and this estimate is propagated ahead to the next time instant according to the usual moving-horizon strategy. Such an estimation criterion ensures robustness to outliers of sufficiently large amplitude.

The paper is organized as follows. In Section 2, the proposed MHE method is described. Stability and robustness properties are illustrated in Sections 3 and 4, respectively. In Section 5, simulation results are presented and discussed. Finally, the conclusions are drawn in Section 6.

Let $\mathbb{N} := \{0, 1, 2, \ldots\}$. The minimum and maximum eigenvalues of a real, symmetric matrix P are denoted by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively; in addition, $P > 0$ means that it is positive definite. Given a generic matrix M, $\|M\| := \left(\lambda_{\max}(M^\top M)\right)^{1/2} = \left(\lambda_{\max}(MM^\top)\right)^{1/2}$. For a vector v, $\|v\| := (v^\top v)^{1/2}$ denotes its Euclidean norm, and $B(r) := \{v \in \mathbb{R}^n : \|v\| \le r\}$ for $r > 0$. Given a sequence of vectors $v_i, v_{i+1}, \ldots, v_j$ for $i < j$, let us define $v_i^j := \mathrm{col}\left(v_i, v_{i+1}, \ldots, v_j\right)$. Moreover, $v_i^j\big|_k$ denotes v_i^j without the kth element, with $k = 1, 2, \ldots, j-i+1$. In other words, $v_i^j\big|_1 := v_{i+1}^j$, $v_i^j\big|_k := \mathrm{col}\left(v_i, v_{i+1}, \ldots, v_{i+k-2}, v_{i+k}, \ldots, v_j\right)$ for $k = 2, 3, \ldots, j-i$, and $v_i^j\big|_{j-i+1} := v_i^{j-1}$. For the sake of simplicity, let $v_i^j\big|_0 := v_i^j$. Finally, recall the square sum bound, namely, given m scalars $s_1, s_2, \ldots, s_m \in \mathbb{R}$, we have

$$\left(\sum_{i=1}^{m} s_i\right)^2 \le m \sum_{i=1}^{m} s_i^2.$$

2. MHE with measures corrupted by outliers

Let us consider the discrete-time linear system

$$x_{t+1} = A x_t + B u_t + w_t \quad \text{(1a)}$$

$$y_t = C x_t + v_t \quad \text{(1b)}$$

where $t = 0, 1, \ldots$ is the time instant, $x_t \in \mathbb{R}^n$ is the state vector, $u_t \in \mathbb{R}^m$ is the control vector, $w_t \in \mathbb{R}^n$ is the system noise vector, $y_t \in \mathbb{R}$ is the measure, and $v_t \in \mathbb{R}$ is the measurement noise.

As to the system disturbance, w_t is supposed to be "small" as compared with the dynamics (i.e., bounded and taking zero or around zero values). In other words, we assume the following.

Assumption 1. There exists $r_w \in (0, \infty)$ such that, for all $t = 0, 1, \ldots$, $\|w_t\| \le r_w$.

The measurement noise, instead, is "small" except on rare occurrences. More specifically, we assume the following.

Assumption 2. There exist $r_v \in (0, \infty)$, $M > r_v$, and a nonnegative, strictly increasing sequence $\{\bar{t}_i\}$ such that, for all $t = 0, 1, \ldots$ and $i = 0, 1, \ldots$, (a) $|v_t| \le r_v$ for $t \notin \{\bar{t}_i\}$, (b) $|v_{\bar{t}_i}| \in (r_v, M]$.

The assumption above means that the measurement noises may take abnormal but bounded values at certain instants $\bar{t}_i$ since, of course, M is much larger than r_v. Such time instants correspond to the outliers and they are unknown. Indeed, we suppose to know r_w and r_v, and the reader is referred to Milanese and Novara (2004) for an overview of the methods to estimate such parameters together with the underlying model. As will be clearer later, the knowledge of M is not required since it would be sufficient to assume that the outliers, though large, are bounded. In Section 4, a lower bound on the absolute value of outliers will be provided in such way that robustness is ensured for the proposed MHE method.

The moving-horizon approach consists in deriving a state estimate at the current time t by using the information given by $y_{t-N}, y_{t-N+1}, \ldots, y_t, u_{t-N}, u_{t-N+1}, \ldots, u_{t-1}$ with the integer $N \ge 1$. More specifically, we aim to estimate $x_{t-N}, \ldots, x_t$ on the basis of such information and of a "prediction" $\bar{x}_{t-N}$ of the state x_{t-N} at the beginning of the moving window. We denote the estimates of $x_{t-N}, \ldots, x_t$ at time t by $\hat{x}_{t-N|t}, \hat{x}_{t-N+1|t}, \ldots, \hat{x}_{t|t}$, respectively.

As compared with the previous literature on MHE, here we consider explicitly the occurrence of outliers in the measures. In such a setting, a natural criterion to derive the estimator consists in resorting to a least-squares approach by explicitly trying to reduce the effect of the outliers. Though in principle we can deal with an arbitrary number of outliers, we restrict our attention to the case of at most only one measurement affected by outlier in the batch of measures included in the sliding window, thus assuming what follows.

Assumption 3. The sequence $\{\bar{t}_i\}$ is such that $\inf_{i\in\mathbb{N}} \left(\bar{t}_{i+1} - \bar{t}_i\right) > N + 1$.

If an outlier corrupts the kth measure of the batch $1, 2, \ldots, N+1$, a least-squares cost function that leaves out such a measure is

$$J_t^k\left(\hat{x}_{t-N}\right) = \mu \,\|\, \hat{x}_{t-N} - \bar{x}_{t-N} \,\|^2 + \alpha_k \sum_{\substack{i=t-N \\ i\neq t-N+k-1}}^{t} \left(y_i - C\hat{x}_i\right)^2 \quad (2)$$

for $k = 1, 2, \ldots, N+1$, where $\mu \ge 0$ and $\alpha_k > 0$. The cost (2) is to be minimized together with the constraints

$$\hat{x}_{i+1} = A\hat{x}_i + Bu_i, \quad i = t-N, \ldots, t-1. \quad (3)$$

If no outlier affects the measures of the batch, we may use all of them:

$$J_t^0\left(\hat{x}_{t-N}\right) = \mu\|\hat{x}_{t-N} - \bar{x}_{t-N}\|^2 + \alpha_0 \sum_{i=t-N}^{t} \left(y_i - C\hat{x}_i\right)^2 \quad (4)$$

where $\alpha_0 > 0$. Of course, also the minimization of (4) has to be performed with the constraints (3). In practice, at each time $t = N, N+1, \ldots$ we have to solve $N+2$ problems given by

$$\min_{\substack{\hat{x}\in\mathbb{R}^n \text{ s.t.} \\ (3)\text{ holds}}} J_t^k\left(\hat{x}\right), \quad k = 0, 1, \ldots, N+1$$