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# A decomposition method for large scale MILPs, with performance guarantees and a power system application ☆

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ABSTRACT

Lagrangian duality in mixed integer optimization is a useful framework for problem decomposition and for producing tight lower bounds to the optimal objective. However, in contrast to the convex case, it is generally unable to produce optimal solutions directly. In fact, solutions recovered from the dual may not only be suboptimal, but even infeasible. In this paper we concentrate on large scale mixed-integer programs with a specific structure that appears in a variety of application domains such as power systems and supply chain management. We propose a solution method for these structures, in which the primal problem is modified in a certain way, guaranteeing that the solutions produced by the corresponding dual are feasible for the original unmodified primal problem. The modification is simple to implement and the method is amenable to distributed computation. We also demonstrate that the quality of the solutions recovered using our procedure improves as the problem size increases, making it particularly useful for large scale problem instances for which commercial solvers are inadequate. We illustrate the efficacy of our method with extensive experimentations on a problem stemming from power systems.


## 1. Introduction

In this paper we investigate mixed-integer optimization problems in the form

$$\begin{cases} \underset{x}{\text{minimize}} & \sum_{i\in I} c_i^\top x_i \\ \text{subject to} & \sum_{i\in I} H_i x_i \le b \\ & x_i \in X_i \quad \forall i \in I. \end{cases} \qquad (\mathcal{P})$$

We refer to $b \in \mathbb{R}^m$ as the *resource vector*, and to the sets $X_i$ as the *subsystems*. We assume that each of the sets $X_i$ is a non-empty, compact, mixed-integer polyhedral set that can be written as

$$X_i = \left\{ x \in \mathbb{R}^{r_i} \times \mathbb{Z}^{z_i} \;\middle|\; A_i x \le d_i \right\},$$

with $A_i \in \mathbb{R}^{m_i \times n_i}$ and $d_i \in \mathbb{R}^{m_i}$. We further assume that the problem $\mathcal{P}$ is feasible and that the total number of subsystems $|I|$ is greater than the length $m$ of the resource vector. Our principal interest is in large-scale optimization problems, i.e. those for which $|I| \gg m$, while remaining finite.

Problem $\mathcal{P}$ can be viewed generically as modeling any problem for which a large number of subproblems defined on the domains $X_i$, whose description can include integer variables, are coupled through a small number of complicating constraints $\sum_{i\in I} H_i x_i \le b$. These coupling constraints determine the limits on the available resources to be shared among the subsystems. Simple examples of problems in this form include classical combinatorial programs such as the multidimensional knapsack problem, in which $X_i = \{0, 1\}$, and $c_i \ge 0$, $H_i \ge 0$ (Wilbaut, Hanafi, & Salhi, 2008).

More complicated instances of problems in the form $\mathcal{P}$, with more detailed models for the subsystems $X_i$, arise in a variety of contexts. In power systems, scheduling the operation of power generation plants (Yamin, 2004) is a decision problem in which the subsystems are the generating units, integer variables in the

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local models arise due to, e.g., start-up and shut-down costs, and the coupling constraints are related to the requirement that generation must match load. In supply chain management, models fitting $\mathcal{P}$ appear in the problem of partial shipments (Dawande, Gavirneni, & Tayur, 2006; Vujanic, Esfahani, Goulart, & Morari, in press-b). Portfolio optimization for small investors, for which mixed-integer models have been proposed, is another example application (Baumann & Trautmann, 2013). Finally, some sparse problems that do not naturally possess the structure of $\mathcal{P}$ can be reformulated to fit our framework by appropriately permuting rows and columns of the constraints matrix; Bergner et al. (2011) propose a method to automate this procedure.

A direct solution of $\mathcal{P}$ is typically problematic when the problem is very large, since the problem amounts to a mixed-integer linear program of possibly very large size. As a result, the Lagrange dual of $\mathcal{P}$ is often taken as a useful alternative, because the resulting dual problem is separable in the subsystems despite the presence of the complicating constraints. When this dual problem is solved by an iterative method, e.g. using the subgradient method (Bertsekas, 1999), a candidate (primal) solution to $\mathcal{P}$ can be computed at each iteration.

For problems affected by non-zero duality gap such as $\mathcal{P}$, however, this approach suffers from a major drawback. Namely, any guarantee about the properties of these candidate solutions is lost. Even at the dual optimal solution, the associated candidate primal solutions may be suboptimal and can even be infeasible.

The principal goal of this paper is to propose a new solution method for problem $\mathcal{P}$ that preserves the attractive features of solution via the Lagrange dual, while at the same time protecting the recovered primal solutions from infeasibility.

*Literature.* Lagrangian relaxation for mixed integer programs was first introduced by Held and Karp (1970), and many of its theoretical properties were described in Geoffrion (1974). Properties of the inner solutions in the convex case are well known (Rockafellar, 1997, Thm. 28.1). It is also well known that in general these properties are lost in the mixed-integer case (Bertsekas, 1999, Section 5.5.3). Because of this, primal recovery methods based on Lagrangian duality are often two-phase schemes in which an infeasible solution is found through duality in the first stage, and in the second stage it is rectified into a feasible one using heuristics, see, e.g., Bertsekas, Lauer, Sandell, and Posbergh (1983) and Redondo and Conejo (1999).

Duality for problems specifically in the form $\mathcal{P}$ has been studied at least as early as in Aubin and Ekeland (1976), where some of its special features were first characterized. In particular, it was noted that the duality gap for this program structure decreases in relative terms as the problem increases in size, as measured by the cardinality of $I$. We will show that the mechanism behind this vanishing gap effect can also be used to recover "good" primal solutions for the mixed-integer program $\mathcal{P}$ directly from the dual, in a way that resembles the convex (zero gap) case.

In practical applications, this behavior of the duality gap has been observed in Bertsekas et al. (1983) in the context of unit commitments for power systems. In this case it is exploited in an algorithm that provides solutions to the extended master problem, but no connection to the solutions of the inner problem is provided. It also appears in the multistage stochastic integer programming literature (Birge & Dempstert, 1996; Caroe & Schultz, 1999), where it is used to gauge the strength of the Lagrangian relaxation, but in which no relations to primal solutions are drawn. Another domain in which diminishing gap has been used is in communications, more precisely in optimization of multicarrier communication systems (Yu & Lui, 2006). However, in this case non-convexity is in the objective function rather than due to the presence of integer variables.

*Current contribution.* In this paper we further investigate duality for programs structured as $\mathcal{P}$ and focus on the primal solutions recovered at the dual optimum.

- We provide a new relation between the optimizers of a convexified form of $\mathcal{P}$ and the solutions obtained from the dual problem. This relation holds under mild conditions that are commonly satisfied in practice.
- Using this relation we can bound the magnitude of the constraint violations of the solutions recovered from the dual. In light of this bound, we propose a new solution method which is guaranteed to produce feasible solutions for $\mathcal{P}$. The method is based on an appropriate contraction of the resources $b$.
- We also provide a performance bound of the solutions recovered, which indicates that their quality improves as the problem size increases. For particular structures, arising e.g. from underlying physical networks, we refine our theoretical results to improve the performance of the method.

From a practical point of view, we note that our proposed procedure is straightforward to implement and is amenable to distributed computation. The performance bound indicates that the method is particularly attractive for large problem instances, for which generic purpose solvers may be inadequate. We show that the theoretical results are effective in practice via extensive numerical experiments on difficult problems stemming from the field of power systems control. Our method substantially outperforms commercial solvers on these problems. The limitations of the proposed method, as well as ideas to mitigate them, are also discussed in the paper.

*Structure of the paper.* The paper is structured as follows: in Section 2 we review some of the known results concerning duality for the specific structure of $\mathcal{P}$, and we provide a new result related to the primal solutions recovered from the dual. In Section 3 we propose a new method for primal solution recovery, and provide performance bounds for these solutions. We also give some results on how to further improve the solutions' quality in some special cases. In Section 4 we verify the efficacy of our proposed method on a difficult optimization problem stemming from power systems, and in Section 5 we conclude the paper.

*Notation.* Given some optimization problem $\mathcal{A}$, we denote with $J^\star_{\mathcal{A}}$ its optimal objective and with $J_{\mathcal{A}}(x)$ the performance of the solution $x$ with respect to the objective of $\mathcal{A}$. For a given set $X$, we denote by $\text{conv}(X)$ its convex hull and by $\text{vert}(X)$ the set of vertices of $\text{conv}(X)$. With "$\geq$" we always intend component-wise inequalities (between vectors or matrices), and with $\otimes$ we indicate the cartesian product of sets. The support of a vector $\text{supp}(x)$ is the set of indexes of the non-zero elements: $\text{supp}(x) = \{i : x_i \neq 0\}$, while $(x)^+$ is the projection of $x$ onto the positive orthant, i.e., $(x)^+ \doteq \max(0, x)$. For the specific structure of $\mathcal{P}$, we use the *overbar* symbol to indicate quantities related to the contracted version of $\mathcal{P}$, as introduced in Section 3. Thus, for instance, $\overline{\mathcal{P}}$ is the contracted form of $\mathcal{P}$ and $\overline{\mathcal{D}}$ is its dual. We use parenthesis to avoid confusing the sub- and superscripts, e.g., we denote by $(x_{\mathcal{P}})_i$ the part of $x_{\mathcal{P}}$ related to subproblem $i \in I$ of problem $\mathcal{P}$. Finally, we use the superscript $H^k$ to denote the $k$th row of matrix $H$.

## 2. Duality for problem $\mathcal{P}$

Consider the dual function $d : \mathbb{R}^m \to \mathbb{R}$ of problem $\mathcal{P}$, defined as

$$d(\lambda) \doteq \min_{x \in X} \left( \sum_{i \in I} c_i^\top x_i + \lambda^\top \left( \sum_{i \in I} H_i x_i - b \right) \right),$$

and then associate to this function the optimization problem

$$\begin{cases} \sup\limits_{\lambda} & -\lambda^\top b + \sum\limits_{i \in I} \min\limits_{x_i \in X_i} \left( c_i^\top x_i + \lambda^\top H_i x_i \right) \\ \text{s.t.} & \lambda \geq 0. \end{cases} \qquad (\mathcal{D})$$