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Robust stability and performance analysis of 2D mixed continuous–discrete-time systems with uncertainty☆

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ABSTRACT

This paper investigates 2D mixed continuous–discrete-time systems whose coefficients are polynomial functions of an uncertain vector constrained into a semialgebraic set. It is shown that a nonconservative linear matrix inequality (LMI) condition for ensuring robust stability can be obtained by introducing complex Lyapunov functions depending polynomially on the uncertain vector and a frequency. Moreover, it is shown that nonconservative LMI conditions for establishing upper bounds of the robust $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms can be obtained by introducing analogous Lyapunov functions depending rationally on the frequency. Some numerical examples illustrate the proposed methodology.

1. Introduction

The study of 2D mixed continuous–discrete-time systems has a long history, with some early works such as Fornasini and Marchesini (1978) and Roesser (1975) introducing basic models, systems theory and stability properties. Applications of these systems can be found in repetitive processes (Rogers & H, 1992), disturbance propagation in vehicle platoons (Fornasini & Valcher, 1997), and irrigation channels (Knorn & Middleton, 2013).

Researchers have investigated several fundamental properties of 2D mixed continuous–discrete-time systems, in particular stability, for which key contributions include Bouagada and Van Dooren (2013), Chesi and Middleton (2014a), Galkowski, Paszke, Rogers, Xu, and Lam (2003), Kar and Singh (2003), and Rogers and H (1992). Other fundamental properties that have been investigated in 2D mixed continuous–discrete-time systems are the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms, for which the contributions include Chesi and Middleton (2015), Paszke, Galkowski, Rogers, and Lam (2008) and Paszke, Rogers, and Galkowski (2011) where conditions based on linear matrix inequalities (LMIs) have been provided. The reader is also referred to Li, Gao, and Wang (2012) and Li, Lam, Gao, and Gu (2015) for related contributions in other areas of 2D systems.

However, these conditions cannot be used whenever the matrices of the model are affected by uncertainty. In fact, in such a case, one should repeat the existing conditions addressing the uncertainty-free case for all the admissible values of the uncertainty. Clearly, this is impossible since the number of values in a continuous set is infinite and one cannot just consider a finite subset of values such as the vertices in the case this set is a polytope. It should be mentioned that various methods have been proposed in the literature for stability and performance analysis of 1D systems affected by uncertainty, such as Aguirre, Garcia-Sosa, Leyva, Solis-Daun, and Carrillo (2015), Aguirre, Ibarra, and Suarez (2002), Bliman (2004), Chesi (2005, 2013), Oliveira and Peres (2007) and Scherer and Hol (2006).

This paper investigates 2D mixed continuous–discrete-time systems affected by uncertainty. It is supposed that the coefficients of the systems are polynomial functions of an uncertain vector constrained into a semialgebraic set. It is shown that an LMI condition for ensuring robust stability can be obtained by introducing complex Lyapunov functions depending polynomially on the uncertain vector and a frequency. Moreover, it is shown that LMI conditions for establishing upper bounds of the robust $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms can be obtained by introducing analogous Lyapunov functions depending rationally on the frequency. These LMI conditions are sufficient for any chosen degree of the complex

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Lyapunov functions, and also necessary for a sufficiently large degree of these functions under some mild assumptions on the set of admissible uncertainties. The LMI conditions proposed in this paper exploit Putinar's Positivstellensatz (Putinar, 1993), which allows one to investigate positivity of a polynomial over a semialgebraic set by using polynomials that can be written as sums of squares of polynomials (SOS). Some numerical examples illustrate the proposed methodology.

This paper extends the preliminary conference papers (Chesi, 2014; Chesi & Middleton, 2014b) by showing that the LMI condition for determining the robust $\mathcal{H}_\infty$ norm is nonconservative (Theorem 3) and by adding the investigation of the robust $\mathcal{H}_2$ norm (Section 5).

The paper is organized as follows. Section 2 provides the problem formulation and some preliminaries about SOS matrix polynomials. Section 3 investigates the robust exponential stability. Section 4 addresses the robust $\mathcal{H}_\infty$ norm. Section 5 addresses the robust $\mathcal{H}_2$ norm. Section 6 presents some illustrative examples. Section 7 concludes the paper with some final remarks. Lastly, the appendices report some additional results.

2. Preliminaries

2.1. Problem formulation

The notation is as follows. The real and complex number sets are denoted by $\mathbb{R}$ and $\mathbb{C}$. The imaginary unit is j. The symbol I denotes the identity matrix (of size specified by the context). The notations $Re(\cdot)$, $Im(\cdot)$ and $|\cdot|$ denote the real part, imaginary part and magnitude. The Euclidean norm and the $\mathcal{L}_2$ norm are denoted by $\|\cdot\|_2$ and $\|\cdot\|_{\mathcal{L}_2}$. The adjoint, determinant, null space and trace are denoted by $adj(\cdot)$, $\det(\cdot)$, $\ker(\cdot)$ and $trace(\cdot)$. The sign is denoted by $sgn(\cdot)$. The notation $A \otimes B$ denotes the Kronecker product of A and B. The complex conjugate, transpose and complex conjugate transpose of A are denoted by $\bar{A}$, A^T and A^H. We say that a complex matrix A is Hermitian if $A^H = A$. The symbol $\star$ denotes corresponding blocks in Hermitian matrices. The notations $A > 0$ and $A \geq 0$ denote Hermitian positive definite and Hermitian positive semidefinite matrix A. The degree is denoted by $\deg(\cdot)$.

Let us consider the 2D mixed continuous–discrete-time system with uncertainty described by

$$\begin{cases} \dfrac{d}{dt}x_c(t,k) = A_{cc}(p)x_c(t,k) + A_{cd}(p)x_d(t,k) \\ \qquad + B_c(p)u(t,k) \\ x_d(t,k+1) = A_{dc}(p)x_c(t,k) + A_{dd}(p)x_d(t,k) \\ \qquad + B_d(p)u(t,k) \\ y(t,k) = C_c(p)x_c(t,k) + C_d(p)x_d(t,k) \\ \qquad + D(p)u(t,k) \end{cases} \tag{1}$$

where $x_c \in \mathbb{R}^{n_c}$ and $x_d \in \mathbb{R}^{n_d}$ are the continuous and discrete states, the scalars $t, k \in \mathbb{R}$ are independent variables, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_y}$ are the input and output, $p \in \mathbb{R}^q$ is a time-invariant uncertain vector, and the matrices $A_{cc} : \mathbb{R}^q \to \mathbb{R}^{n_c \times n_c}$, $A_{cd} : \mathbb{R}^q \to \mathbb{R}^{n_c \times n_d}$, $A_{dc} : \mathbb{R}^q \to \mathbb{R}^{n_d \times n_c}$, $A_{dd} : \mathbb{R}^q \to \mathbb{R}^{n_d \times n_d}$, $B_c : \mathbb{R}^q \to \mathbb{R}^{n_c \times n_u}$, $B_d : \mathbb{R}^q \to \mathbb{R}^{n_d \times n_u}$, $C_c : \mathbb{R}^q \to \mathbb{R}^{n_y \times n_c}$, $C_d : \mathbb{R}^q \to \mathbb{R}^{n_y \times n_d}$ and $D : \mathbb{R}^q \to \mathbb{R}^{n_y \times n_u}$ are polynomial functions of degree not greater than d_{sys}.

It is supposed that p is constrained as

$$p \in \mathcal{P} \tag{2}$$

where $\mathcal{P}$ is the semialgebraic set

$$\mathcal{P} = \left\{ p \in \mathbb{R}^q : a_i(p) \geq 0 \ \forall i = 1 \dots, n_a \right\} \tag{3}$$

and $a_i(p) i = 1, \dots, n_a$, are polynomials. No assumption is introduced on these polynomials at this stage except that $\mathcal{P}$ must be nonempty (further assumptions will be introduced on these polynomials with Definition 2 in Section 3, which will be exploited in Theorems 2, 3 and 5). Let us observe that $\mathcal{P}$ can represent a number of sets typically used to model uncertain systems, for instance:

(1) hyper-ellipsoids of the form $\{p \in \mathbb{R}^q : p^T A p \leq 1\}$ where $A > 0$ by choosing $n_a = 1$ and $a_1(p) = 1 - p^T A p$;
(2) hyper-rectangles of the form $\{p \in \mathbb{R}^q : p_i \in [p_i^-, p_i^+],\ i = 1, \dots, q\}$ where $p_i^-, p_i^+ \in \mathbb{R}$, by choosing $n_a = q$ and $a_i(p) = (p_i^- - p_i)(p_i - p_i^+)$;
(3) polytopes of the form $\{p \in \mathbb{R}^q : v_i^T p \leq w_i,\ i = 1, \dots, l\}$ where $v_i \in \mathbb{R}^q$ and $w_i \in \mathbb{R}$, by choosing $n_a = l$ and $a_i(p) = w_i - v_i^T p$.

Extending the classical definition of exponential stability of 2D mixed continuous–discrete-time systems (Pandolfi, 1984), we say that the system (1)–(3) is robustly exponentially stable if, for a null input $u(t,k)$, there exist $\beta > 0$ and $\gamma > 0$ such that

$$\left\| \begin{pmatrix} x_c(t,k) \\ x_d(t,k) \end{pmatrix} \right\|_2 \leq \beta \varrho e^{-\gamma \min\{t,k\}} \tag{4}$$

for all $t \geq 0$ and $k \geq 0$, for all initial conditions $x_c(0,k)$ and $x_d(t,0)$, and for all $p \in \mathcal{P}$, where

$$\begin{cases} \varrho = \max\{\varrho_c, \varrho_d\} \\ \varrho_c = \sup_{k \geq 0} \|x_c(0,k)\|_2 \\ \varrho_d = \sup_{t \geq 0} \|x_d(t,0)\|_2 . \end{cases} \tag{5}$$

Problem 1. The first problem addressed in this paper consists of establishing whether (1)–(3) is robustly exponentially stable. □

Next, let us introduce the robust $\mathcal{H}_\infty$ norm of (1)–(3) as

$$\gamma_\infty^* = \sup_{p \in \mathcal{P}} \gamma_\infty(p) \tag{6}$$

where $\gamma_\infty(p)$ is the $\mathcal{H}_\infty$ norm of (1) for the fixed value p of the uncertainty given by

$$\gamma_\infty(p) = \sup_{u:\ \|u\|_{\mathcal{L}_2} \neq 0} \frac{\|y\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}} \tag{7}$$

and $\|\cdot\|_{\mathcal{L}_2}$ is the $\mathcal{L}_2$ norm defined as

$$\|u\|_{\mathcal{L}_2} = \sqrt{\sum_{k=0}^{\infty} \int_0^\infty \|u(t,k)\|_2^2 dt}. \tag{8}$$

Problem 2. The second problem addressed in this paper consists of determining the robust $\mathcal{H}_\infty$ norm of (1)–(3), i.e., γ_∞^*. □

Lastly, let us introduce the robust $\mathcal{H}_2$ norm of (1)–(3) as

$$\gamma_2^* = \sup_{p \in \mathcal{P}} \gamma_2(p) \tag{9}$$

where $\gamma_2(p)$ is the $\mathcal{H}_2$ norm of (1) for the fixed value p of the uncertainty given by

$$\gamma_2(p) = \sqrt{\sum_{l=1}^{n_u} \sum_{k=0}^{\infty} \int_0^\infty g^T(t,k,l) g(t,k,l) dt} \tag{10}$$

where $g(t,k,l)$ is the impulse response due to a Dirac impulse applied at $k = 0$ to the lth channel, i.e., the solution of $y(t,k)$ for null initial conditions and $u(t,k)$ given by

$$u(t,k) = \begin{cases} \delta(t) b(l) & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

where $\delta(t)$ is the Dirac impulse and $b(l)$ is the lth canonical basis vector in $\mathbb{R}^{n_u}$.

Problem 3. The third problem addressed in this paper consists of determining the robust $\mathcal{H}_2$ norm of (1)–(3), i.e., γ_2^*. □