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Nonlinear state estimation on unit spheres using manifold particle filtering

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ABSTRACT

In many applications in engineering, one is interested in tracking a dynamic system whose state evolves on a manifold. Solutions to such problems frequently must resort to nonlinear filtering techniques as many manifolds can be described as equality restrictions on higher-dimensional embedding spaces. We propose in this paper a new particle filtering (PF) method to track the states of dynamic systems that evolve according to a random walk on the unit sphere. We derive an approximation to the intractable optimal importance function and develop a Markov Chain Monte Carlo (MCMC) method to sample from it. The system state variable is then estimated via a Monte Carlo approximation of its intrinsic mean on the sphere, obtained from the Karcher mean of the particle set. As we verify via computer simulations, the proposed method shows improved performance compared to previous Constrained Extended Kalman filters and Bootstrap PF solutions.

1. Introduction

In many engineering problems, it is necessary to track the hidden state of a dynamic system from available observed data. Conventional tracking algorithms found in the literature such as standard Kalman filters or particle filters normally assume that the hidden state variables evolve on a linear Euclidean space, e.g. $\mathbb{R}^L$, where L is the dimension of the state vector. However, in several real-world applications, e.g. attitude or pose estimation [1,2] in navigation or robotics, communication channel equalization [3], and tracking problems using images [4] or microphone arrays [5], the underlying state vector is physically constrained to lie instead on a manifold, causing the aforementioned conventional tracking algorithms to underperform.

Previous work sought to address the problem of manifold-constrained state vectors by proposing modifications to standard tracking algorithms. Constrained Kalman filters [6,7] for example can enforce quadratic constraints on the state variables, being able to deal with the case in which the states evolve on a unit sphere, but they tend to perform poorly with nonlinear observation models.

Particle Filters [2], which are better suited for nonlinear data models, can, on the other hand, be also formulated in more general manifolds. For example, Ref. [8] considers a problem in which the hidden states evolve on a matrix Lie group. In [1], a particle filtering algorithm uses a mixture Bingham model to approximate the posterior distribution of unit-norm quaternions. In [5] and [9], bootstrap particle filters employing a blind (i.e. data-independent) importance function were derived for a model in which the state evolves on a unit sphere according to a Von Mises–Fisher random walk. Finally, in [3], we considered both supervised and unsupervised estimation of a time-invariant, unit-norm digital communication channel assuming a linear observation model at the receiver's end.

In this paper, we introduce a novel particle filter algorithm to track a time-variant dynamic state vector on the unit hypersphere $\mathbb{S}^{L-1}$ embedded in $\mathbb{R}^L$, where L is an arbitrary integer number. Our main original contributions in this work are twofold. First, we derive a new Fisher–Bingham (FB) approximation to the intractable optimal importance function on the unit hypersphere. The proposed approximation extends previous results in [3] to a more general model with time-varying states and an arbitrary nonlinear observation equation. Second, we propose an iterative Markov Chain Monte Carlo (MCMC) method to sample from the approximated importance function on $\mathbb{S}^{L-1}$. As we show in the paper, the proposed Fisher–Bingham parametric approximation to the optimal importance function is the equivalent on $\mathbb{S}^{L-1}$ to the Gaussian approximation to the optimal importance function introduced in [10] for particle filters specified in $\mathbb{R}^L$ with arbitrary nonlinear observation models. This equivalence comes from the fact that the

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Fisher–Bingham density is the result of conditioning a Gaussian random vector to lie on the unit sphere [11].

The remainder of this paper is organized as follows: in Sec. 2 we introduce the problem under consideration and, in Sec. 3, we formulate its solution via particle filters on manifolds. In Sec. 4, we derive the FB approximation to the optimal importance function, describe the MCMC algorithm to sample from it, show the corresponding analytical expression to update the particle importance weights at each time instant, and introduce a method to estimate the hidden state vector from the weighted importance function samples by their *Karcher mean* [12], which, as we argue in the paper, are, from a differential geometric point of view, the equivalent on $\mathbb{S}^{L-1}$ to the minimum-mean-square-error (MMSE) estimate on $\mathbb{R}^L$. In Section 5, we particularize the proposed method to the solution of the problem of supervised identification of frequency-selective communication channels and show via Monte Carlo simulation that our novel PF outperforms both the bootstrap PF in [5,9] and a standard constrained extended Kalman filter (EKF) when a strongly nonlinear amplifier is assumed at the receiver. Finally, we present the conclusions of our work in Sec. 6.

2. Problem setup

In the remainder of the paper, we use normal lowercase letters to denote scalars, bold lowercase letters to denote vectors, and uppercase letters to denote matrices. The distinction between real, random variables, and samples of random variables is implied in context.

Let $\boldsymbol{s}_n \in \mathbb{S}^{L-1}$ denote the state of a time-varying system at time instant n and $\mathbb{S}^{L-1} = \{\boldsymbol{s} \in \mathbb{R}^L : \|\boldsymbol{s}\| = 1\}$ the unit L-sphere. The state $\boldsymbol{s}_n$ is supposed to evolve according to a Von Mises–Fisher random walk on the sphere, i.e.,

$$\boldsymbol{s}_n|\boldsymbol{s}_{n-1} \sim \mathrm{vMF}(\boldsymbol{s}_n|\kappa;\boldsymbol{s}_{n-1}),\ n > 0, \tag{1}$$

with[1] $p(\boldsymbol{s}_0) \propto 1$, where

$$\mathrm{vMF}(\boldsymbol{s}|\kappa;\boldsymbol{a}) \triangleq \frac{1}{c_{vMF}(\kappa, L)} \exp\left\{\kappa \boldsymbol{a}^T \boldsymbol{s}\right\} \tag{2}$$

indicates a Von Mises–Fisher (vMF) [13] probability density function (p.d.f.), $\boldsymbol{s}$, $\boldsymbol{a} \in \mathbb{S}^{L-1}$, $\kappa \in \mathbb{R}^+$ is a known hyperparameter, and c_{vMF} is the vMF p.d.f. normalization constant, given by

$$c_{vMF}(\kappa, L) = \frac{\kappa^{L/2-1}}{(2\pi)^{L/2} J_{L/2-1}(\kappa)}, \tag{3}$$

where J_ν denotes the modified Bessel function of the first kind and order ν.

The system states give rise to the observation sequence

$$y_n = g(\boldsymbol{\theta}_n^T \boldsymbol{s}_n) + v_n,\ n > 0, \tag{4}$$

where $\boldsymbol{\theta}_n \in \mathbb{R}^L$ is a *known* parameter vector, $g(\cdot)$ denotes a possibly nonlinear function, and v_n is a sequence of zero-mean, independent Gaussian random variables of variance σ^2.

Considering the model described by (1)–(4), we aim at recursively determining the filtering density $p(\boldsymbol{s}_n|y_{1:n})$, where $y_{1:n} \triangleq \{y_1, \ldots, y_n\}$. It can be verified that even in the case in which $g(\cdot)$ is linear, there is no known closed form to the sought filtering density, which prompted us to resort to particle filters.

[1] We denote by $p(\cdot)$ the probability density function of a random variable or vector.

Algorithm 1 Bootstrap Particle Filter.

for $n > 0$ **do**
  for $p = 1 : P$ **do**
    •Draw $\boldsymbol{s}_n^{(q)} \sim \mathrm{vMF}(\boldsymbol{s}_n|\kappa;\boldsymbol{s}_{n-1})$.
    •Update $w_n^{(q)}$ via (8).
  end for
  •Normalize the weights, i.e., $\sum_{q=1}^{Q} w_n^{(q)} = 1$.
  •Estimate $\check{\boldsymbol{s}}_n$ iterating (28) until $\|\check{\boldsymbol{s}}_n(i+1) - \check{\boldsymbol{s}}_n(i)\|$ is sufficiently small.
  •Resample the particle set $\{\boldsymbol{s}_n^{(q)}, w_n^{(q)}\}_{q=1}^{Q}$.
end for

3. Particle filters

Particle filters [14] (PF) are well-established techniques for approximating filtering densities for nonlinear or non-Gaussian estimation problems. PF approximate the states' posterior probabilities by the weighted sum

$$\mathrm{Prob}(\{\boldsymbol{s}_{0:n} \in G\}|y_{1:n}) \approx \sum_{q=1}^{Q} w_n^{(q)} \delta_{\boldsymbol{s}_{0:n}^{(q)}}(G), \tag{5}$$

where G is a subset of $\underbrace{\mathbb{S}^{L-1} \times \cdots \times \mathbb{S}^{L-1}}_{(n+1)\text{ times}}$, $\delta_x(G)$ is a Dirac measure, defined as 1 if $x \in G$ and 0 otherwise, $\boldsymbol{s}_{0:n}^{(q)}$ are the so-called particles, sequentially sampled from the *importance function* $\boldsymbol{s}_n^{(q)} \sim \pi(\boldsymbol{s}_n|\boldsymbol{s}_{0:n-1}^{(q)}, y_{1:n})$, $Q \gg 1$ denotes the number of particles, and $w_n^{(q)}$ the particle weights, which obey $\sum_{q=1}^{Q} w_n^{(q)} = 1$ and can be recursively determined as [15]

$$w_n^{(q)} \propto w_{n-1}^{(q)} \frac{p(y_n|\boldsymbol{s}_{0:n}^{(q)}, y_{1:n-1})\, p(\boldsymbol{s}_n^{(q)}|\boldsymbol{s}_{0:n-1}^{(q)}, y_{1:n-1})}{\pi(\boldsymbol{s}_n^{(q)}|\boldsymbol{s}_{0:n-1}^{(q)}, y_{1:n})}. \tag{6}$$

Considering that $\boldsymbol{s}_n$ is Markovian as a consequence of (1) and that the likelihood function (4) only depends on $\boldsymbol{s}_n$, Eq. (6) simplifies to

$$w_n^{(q)} \propto w_{n-1}^{(q)} \frac{p(y_n|\boldsymbol{s}_n^{(q)})\, p(\boldsymbol{s}_n^{(q)}|\boldsymbol{s}_{n-1}^{(q)})}{\pi(\boldsymbol{s}_n^{(q)}|\boldsymbol{s}_{0:n-1}^{(q)}, y_{1:n})}. \tag{7}$$

Choosing the so-called *prior* importance function, i.e., the transition density $\pi(\boldsymbol{s}_n|\boldsymbol{s}_{0:n-1}, y_{1:n}) = p(\boldsymbol{s}_n|\boldsymbol{s}_{n-1}) = \mathrm{vMF}(\boldsymbol{s}_n|\kappa;\boldsymbol{s}_{n-1})$ leads to the algorithm known as *bootstrap filter* (Algorithm 1). In this case, the weights are updated according to

$$w_n^{(q)} \propto w_{n-1}^{(q)}\, p(y_n|\boldsymbol{s}_n^{(q)}). \tag{8}$$

In practice, the weighted particle set $\left\{\boldsymbol{s}_n^{(q)}, w_n^{(q)}\right\}_{q=1}^{Q}$ degenerates [14] after a few iterations unless some measure to restore weight uniformity is performed. To this aim, all particle filters in this work employ a residual resampling [16] step at each iteration.

Comparison to previous work Note that, for a linear function $g(\cdot)$ in (4), the bootstrap particle filter described in Algorithm 1 is a particular instance of the algorithm introduced in [9], as $V_{k,L}$, the Stiefel manifold in $\mathbb{R}^L$, reduces to the unit sphere $\mathbb{S}^{L-1}$ for $k = 1$ [9]. Reference [5], on the other hand, also proposes a bootstrap particle filter on $\mathbb{S}^{L-1}$ that employs a Von Mises–Fisher prior importance function similar to the one used in our Algorithm 1. However, the observations in [5] are also constrained to be on the unit L-sphere whereas, in our work, they lie on the Euclidian space $\mathbb{R}^L$. Due to the aforementioned difference in the observation model, the algorithm in [5] is not directly comparable to the algorithms proposed in this paper.

The main drawback, however, of the filters proposed both in [9] and in [5] is the use of a blind (i.e. data-independent) importance function, which generally leads to a rapid decrease in the effective

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