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# Uncertainty relations for signal concentrations associated with the linear canonical transform

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ABSTRACT

The linear canonical transform (LCT) has been shown to be a useful and powerful tool in optics and signal processing. In this paper, a new uncertainty relation in the LCT domain has been obtained at first. It shows that nonzero signal's energy in two arbitrary LCT domains cannot be arbitrarily large simultaneously, which is the generalization of the uncertainty principle for signal concentrations in the Fourier domain. Meanwhile, the signals which are the best in achieving simultaneous concentration in two arbitrary LCT domains are also proposed. In addition, some potential applications are presented to show the effectiveness of the theorems.



## 1. Introduction

The linear canonical transform (LCT) [1–3], which was introduced in 1970s, is a four parameter class of linear integral transform. The well-known signal processing operations, such as the Fourier transform (FT), the fractional Fourier transform (FRFT), the Fresnel transform (FRT) [2–4] and others are all its special cases. Due to the LCT has extra degrees of freedom, it is more flexible and has shown to be a useful and powerful analyzing tool in signal processing, optics, communications, etc [4–12]. The continuous-time LCT with parameter $M=(a,b,c,d)$ of a signal $f(t)$ is defined as [1]

$$F_{(a,b,c,d)}(u)=L_M[f(t)](u)=\begin{cases}\int_{-\infty}^{\infty} f(t)K_M(u,t)dt, b\neq 0\\ \sqrt{d}e^{j(cd/2)u^2}f(du), b=0\end{cases} \tag{1}$$

where $a,b,c,d$ are real numbers satisfying $ad-bc=1$, and the kernel $K_M(u,t)$ is given by

$$K_M(u,t)=C_M e^{j(\frac{a}{2b}t^2-\frac{1}{b}tu+\frac{d}{2b}u^2)} \tag{2}$$

and

$$C_M=\sqrt{1/j2\pi b} \tag{3}$$

In this paper, we only confine our attention to LCT for $b\neq 0$, because the LCT of a signal is essentially a chirp multiplication when $b=0$. In addition, without loss of generality, we assume $b>0$ in the following sections. Conversely, the inverse LCT is expressed as

$$f(t)=\int_{-\infty}^{\infty} F_{(a,b,c,d)}(u)K_{M^{-1}}(u,t)du \tag{4}$$

where $M^{-1}=(d,-b,-c,a)$.

Simultaneously, the uncertainty principle is an elementary theorem in signal processing and physics [13,14]. As the LCT has recently been found many applications in signal processing, the studies of the uncertainty principle associated with the LCT have blossomed in recent years [15–27]. For example, in [19], Stern extended the classical uncertainty principle to the LCT and derived a tighter lower bound for real signals in the LCT domain. Following Stern's work, Sharma et al. [26], Zhao et al. [23] and Xu et al. [25] generalized the uncertainty principle to two arbitrary LCT domains for real signals independently. In [15,20], the classical uncertainty principle to the LCT for complex signals has been obtained. Shi et al. [16] also addressed the uncertainty principle to the LCT of complex signals via operator methods. The entropic uncertainty relations associated with the LCT have been attained in [22]. In [13], it has presented the uncertainty principles for discrete signals between the time spread and the LCT frequency spread associated with the LCT. Furthermore, Zhang has derived the tighter uncertainty principles for the LCT in terms of matrix decomposition

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in [4]. However, there is one pending issue associated with the uncertainty relations in the LCT domain.

This one relates to the uncertainty principle for signal concentrations in the LCT domain. To the best of our knowledge, there are no results published associated with the uncertainty principle for signal concentrations in the LCT domain. However, in communication theory setting [28,29], it is necessary to know how close one can come to simultaneous limiting in two different domains, and what price is that one has to pay. Hence, a sharp measure of the concentrations of $F_{(a_1,b_1,c_1,d_1)}(u)$ and $F_{(a_2,b_2,c_2,d_2)}(v)$ need to be established. To resolve these issues, more natural and meaningful measure of concentration of a signal is proposed in the following, from the communications point of view.

$$\Delta u^2_{(a_1,b_1,c_1,d_1)} = \frac{\int_{-U}^{+U} \left|F_{(a_1,b_1,c_1,d_1)}(u)\right|^2 du}{\int_{-\infty}^{+\infty} \left|F_{(a_1,b_1,c_1,d_1)}(u)\right|^2 du} \tag{5}$$

where $U > 0$. When substituting special parameters into (5), the sharp measure for signal concentration in the FT domain can be obtained [30]. It states that any nonzero signal cannot have arbitrarily large proportions of energy in both a finite time duration and a finite frequency bandwidth [30,31]. Due to the LCT has many applications in signal processing, it is therefore theoretically interesting and practically useful to explore if a signal can have arbitrarily large fractions of energy in two different LCT domains simultaneously.

In this paper, the uncertainty relations for signal concentrations in the LCT domain by using sharper measures of concentrations defined in (5) have been derived. The rest of this paper is organized as follows. In next section, the preliminaries are proposed. In section 3, we first derive that a nonzero signal cannot have arbitrarily large proportions of energy in two different LCT domains. Then, the signals which are the best in achieving simultaneous concentration in two different LCT domains are also derived. Some potential applications of these results are given in section 4. Section 5 concludes this paper.

## 2. Preliminaries

### 2.1. Some facts for the LCT

The relationship between the LCT and the FT is given in the following, which will be used in this paper [1].

$$\begin{aligned} &F_{(a_2,b_2,c_2,d_2)}(v) \\ &= L_{BA^{-1}}[F_{(a_1,b_1,c_1,d_1)}(u)](v) \\ &= \sqrt{2\pi}C_{A_1}e^{jd_3v^2/2b_3}\Im[F_{(a_1,b_1,c_1,d_1)}(u)e^{ja_3v^2/2b_3}](v/b_3) \end{aligned} \tag{6}$$

where $A = (a_1, b_1, c_1, d_1)$, $B = (a_2, b_2, c_2, d_2)$, $A_1 = BA^{-1} = (a_2d_1 - b_2c_1, b_2a_1 - a_2b_1, c_2d_1 - d_2c_1, a_1d_2 - b_1c_2) = (a_3, b_3, c_3, d_3)$ and $\Im$ denotes the FT operator.

On the other hand, motivated by the fact that the LCT is a generalization of the FT, the ordinary convolution operation associated with two LCT domains can be defined as follows.

$$\begin{aligned} &(F_{(a_1,b_1,c_1,d_1)}\Theta h)(u) \\ &= e^{-ja_3u^2/2b_3}\int_{-\infty}^{+\infty} F_{(a_1,b_1,c_1,d_1)}(u')e^{ja_3u'^2/2b_3}h(u-u')du' \end{aligned} \tag{7}$$

where $\Theta$ denotes the ordinary convolution operation in the LCT domain. Then according to the properties of the LCT, we can easily obtain that

$$L_{BA^{-1}}[(F_{(a_1,b_1,c_1,d_1)}\Theta h)(u)](v) = \sqrt{2\pi}F_{(a_2,b_2,c_2,d_2)}(v)H(v/b_3) \tag{8}$$

where $H(v)$ denotes the FT of $h(t)$. It is easy to verify that if we substitute special parameters into (7), the ordinary convolution of the FT in [32] can be obtained.

### 2.2. Notations

Let H be a Hilbert space with inner product $<\cdot,\cdot>$ given by

$$< f, g >= \int_{-\infty}^{+\infty} f(t)g^*(t)dt \tag{9}$$

Meanwhile, let B(H) denotes the set of all bounded linear operators. If $D(\lambda) \in$ B(H), then the adjoint of the operator $\lambda$, $\lambda^+$ is defined as [33]

$$< \lambda f, g >=< f, \lambda^+ g > \quad \forall f(t) \in D(\lambda), g(t) \in D(\lambda^+) \tag{10}$$

In addition, the $\lambda$ is said to be Hermitian or self-adjoint when $\lambda = \lambda^+$.

## 3. Uncertainty principle for signal concentrations in the LCT domain

In this section, two sets of functions are defined at first in the following, which will be used in this paper.

$$\mathrm{A} = \{F_{(a_1,b_1,c_1,d_1)} : F_{(a_1,b_1,c_1,d_1)}(u) = 0 \qquad \forall |u| > U\} \tag{11}$$

$$\mathrm{B} = \{F_{(a_2,b_2,c_2,d_2)} : F_{(a_2,b_2,c_2,d_2)}(v) = 0 \quad \forall |v| > \Omega\} \tag{12}$$

Then, according to the definitions of (11) and (12), two truncation operators are introduced in the following

$$(\Phi_U F_{(a_1,b_1,c_1,d_1)})(u) = \begin{cases} F_{(a_1,b_1,c_1,d_1)}(u) & |u| < U \\ 0 & |u| \geq U \end{cases} \tag{13}$$

$$\begin{aligned} (\Gamma_\Omega F_{(a_1,b_1,c_1,d_1)})(u) &= L_{BA^{-1}}[(\Phi_u F_{(a_2,b_2,c_2,d_2)})(v)](u) \\ &= \int_{-\Omega}^{+\Omega} F_{(a_2,b_2,c_2,d_2)}(v)K_{BA^{-1}}(v,u)dv \\ &= F_{(a_1,b_1,c_1,d_1)}(u)\Theta\frac{\sin(\Omega u/b_3)}{\pi u} \end{aligned} \tag{14}$$

From (13) and (14), it is easy to obtain that

$$\Phi_U\Phi_U = \Phi_U, \quad \Gamma_\Omega\Gamma_\Omega = \Gamma_\Omega \tag{15}$$

Therefore, $\Phi_U, \Gamma_\Omega$ are self-adjoint operators. Thus, the measure of concentrations of a signal $f(t)$ can be rewritten as

$$\begin{aligned} \Delta u^2_{(a_1,b_1,c_1,d_1)} &= \frac{\left\|(\Phi_U F_{(a_1,b_1,c_1,d_1)})(u)\right\|^2}{\left\|F_{(a_1,b_1,c_1,d_1)}(u)\right\|^2} \\ &= \frac{\langle\Phi_U\rangle_{F_{(a_1,b_1,c_1,d_1)}}}{\left\langle F_{(a_1,b_1,c_1,d_1)}, F_{(a_1,b_1,c_1,d_1)}\right\rangle} \end{aligned} \tag{16}$$

Likewise, we can have

$$\begin{aligned} \Delta v^2_{(a_2,b_2,c_2,d_2)} &= \frac{\left\|(\Gamma_\Omega F_{(a_2,b_2,c_2,d_2)})(v)\right\|^2}{\left\|F_{(a_2,b_2,c_2,d_2)}(v)\right\|^2} \\ &= \frac{\langle\Gamma_\Omega\rangle_{F_{(a_2,b_2,c_2,d_2)}}}{\left\langle F_{(a_2,b_2,c_2,d_2)}, F_{(a_2,b_2,c_2,d_2)}\right\rangle} \end{aligned} \tag{17}$$

Then according to the Parseval's theorem of the LCT [4], we can rewrite (17) as