Digital Signal Processing 81 (2018) 186–197

Contents lists available at ScienceDirect

Digital Signal Processing

www.elsevier.com/locate/dsp

On the asymptotic distribution for the periodograms of almost periodically correlated (cyclostationary) processes

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ARTICLE INFO

Article history:
Available online 27 July 2018

Keywords:
Almost periodically correlated time series
Limiting distribution
Periodogram
Spectral analysis

ABSTRACT

Almost periodically correlated (APC) processes have almost periodic mean and auto-covariance functions. These processes have spectral mass on lines parallel to the diagonal, $T_j(x) = x \pm \alpha_j$, $j = 1, 2, \dots$, in the two-dimensional spectral plane $[0, 2\pi)^2$, and contain stationary and periodically correlated processes. The aim of this article is to make almost periodically correlated processes practical in time series modeling. The main results are on the asymptotic unbiasedness of the periodogram, and the limiting distribution for the finite Fourier transform and the periodogram of APC processes. First, we put light on APC processes in more details. The periodogram is introduced and by using an auxiliary operator, we prove that the limiting distribution of the finite Fourier transform and the periodogram are multivariate complex normal and complex Wishart distributions, respectively. Finally, the accuracy of theoretical results is investigated through simulation study.

1. Introduction

There is a huge literature on the theories and applications of the stationary processes. The support of the spectral measure of these processes is contained on the main diagonal, $T_1(x) = x$, in the bifrequency plane, $[0, 2\pi)^2$. The stationary processes have been studied in many references. For example, see Hannan [19], Anderson [2], Hannan [20], Priestley [41], Yaglom [46], Brockwell and Davis [7], Pourahmadi [42,43], Wu and Pourahmadi [47,48], Xiao and Wu [49], Mahmoudi et al. [35].

These works are dependent on the assumption of stationarity. But this assumption is not adequate for many processes which have periodic rhythm. However, periodically correlated (cyclostationary) processes and almost periodically correlated processes provide natural ways to describe rhythmic random processes. The class of periodically correlated time series which was introduced by Gladyshev [17] has a periodic mean and auto-covariance functions and is applied in various areas of science and technology, such as climatology, hydrology, engineering, signal processing and economics. The support of the spectral measure of these processes with period m is contained in a set of equally spaced lines parallel to the main diagonal, $T_j(x) = x \pm \frac{2\pi j}{m}$, $j = -m+1, \dots, m-1$, in the bifrequency plane $[0, 2\pi)^2$. Processes with almost periodic covariance functions have almost periodic mean and auto-covariance functions. These processes have spectral mass on lines parallel to the diagonal, $T_j(x) = x \pm \alpha_j$, $j = 1, 2, \dots$, in the two-dimensional spectral plane $[0, 2\pi)^2$, and contain stationary and periodically correlated processes. Gladyshev [18], Alekseev [1], Hurd [21], Gardner [12], Hurd [22], Hurd and Leskow [23], Leskow and Weron [31], Dandawate and Giannakis [10], Gardner [13], Gerr and Allen [14,16], Leskow [30], Lii and Rosenblatt [32,33], Gardner et al. [15], Hurd and Miamee [24], Soltani and Azimmohseni [44], Lenart [25, 26], Napolitano [38], Mahmoudi et al. [36], Mahmoudi and Maleki [34], Nematollahi et al. [39] studied the periodically correlated (PC, in short) and almost periodically correlated (APC, in short) processes.

The asymptotic distribution of the periodogram is important in estimation for the spectral density. The asymptotic distribution of the periodograms of the stationary processes can be found in many text books of time series. The general approach is to apply a factorization of the spectral density. It has been more common to employ the square root factorization, which will lead to a moving average representation, in terms of an *i.i.d.* sequence and square absolute summable coefficients. In Hannan [19] the asymptotic results are derived under the continuity of the spectral density. The spectral factorization was applied by Brillinger [5] for introducing principal components and other multivariate techniques in spectral domain for multiple time series. The work

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https://doi.org/10.1016/j.dsp.2018.07.004
1051-2004/

of Brillinger [5] relies on the existence and absolute summability of the joint cumulant functions, forcing the spectral density to be continuous and bounded. Brillinger [6] indicated a mixing condition under which a net of Fourier transforms, of a stationary generalized process over an abelian locally compact group, had a limiting normal distribution. The asymptotic results in Brockwell and Davis [7] are based on this factorization. Cholesky decomposition of the spectral density of the underlying multivariate stationary process appears to be useful in spectral analysis and prediction. The work of Dai and Guo [9] on smoothing the spectral density estimations of the multivariate stationary processes is based on the Cholesky decomposition. For PC processes, Miamee and Soltani [37] applied the Cholesky decomposition for the spectral representations. Soltani and Azimmohseni [44] applied these decompositions to derive the asymptotic distribution of the periodograms of time series data sampled from a discrete time PC process. Peligrad and Wu [40] considered asymptotic behavior of Fourier transforms of stationary ergodic sequences with finite second moments and established a central limit theorem (CLT) for almost all frequencies and also an annealed CLT. Lenart [27] assumed that the considered APC time series is α-mixing with corresponding mixing sequence denoted by $\alpha(\cdot)$. Concerning the limiting distribution, he stated that, for any frequency $\lambda \in [0, 2\pi)$, the limiting distribution of the discrete Fourier transform for any subsample of $\{X_0, \ldots, X_{N-1}\}$, is bivariate normal (Theorem 2.1 and 2.2 in Lenart [27], page 255). Lenart and Pipien [28] generalized the theorems presented in Lenart [27]. As Lenart [27], they assumed α-mixing APC time series and proved that, for any set of frequencies $\{\lambda_1, \ldots, \lambda_p\} \in [0, 2\pi)$, the limiting joint distribution of the discrete Fourier transform for any subsample of $\{X_0, \ldots, X_{N-1}\}$, is $2p$-variate normal (Theorem 2.1 in Lenart and Pipien [28], page 89). In other words, they considered a subsample of $\{X_0, \ldots, X_{N-1}\}$, and computed the limiting joint distribution of the discrete Fourier transform for frequency vector $(\lambda_1, \ldots, \lambda_p)^T$. Lenart and Pipien [29] indicated that under α-mixing condition, for any frequency $\lambda \in [0, 2\pi)$, the limiting joint distribution of the discrete Fourier transform for multivariate time series of $\{X_0, \ldots, X_{N-1}\}$, is $2r$-variate normal (Theorem 1 in Lenart and Pipien [29], page 211). In other words, they fixed $\lambda \in [0, 2\pi)$, and computed the limiting joint distribution of the discrete Fourier transform. Azimmohseni et al. [3] introduced a weighted periodogram in the class of smoothed periodograms as a consistent estimator for the spectral density matrix of a PC process and derive its limiting distribution as a certain finite linear combination of Wishart distribution.

In this work, we are concerned with the asymptotic distribution of the periodogram. Concerning the spectral support reconditions, we introduce a procedure to visualize the spectral support image of APC processes. Concerning the limiting distribution, we prove that the limiting distribution of the fast Fourier transform and the periodogram of APC processes are multivariate normal and Wisharts distributions, respectively.

The rest of the paper is organized as follows. In Section 2, we provide notations, and preliminaries. In Section 3, our auxiliary operator is defined and its properties are investigated. Main theoretical results of the paper are given in Section 4. In Section 5, we give examples and do inference using real and simulated data.

2. Almost periodically processes

In this section, we recall the basic definitions and introduce the notation which is helpful for subsequent work. Let $(D, \mathcal{D})$, $D = [0, 2\pi)$, be a measurable space and let $L^2(\Omega, \mathcal{F}, P)$ be the Hilbert space of complex random variables X on the probability space $(\Omega, \mathcal{F}, P)$ with finite second moments, $E|X|^2 < \infty$.

Definition 2.1 *(Corduneanu [8]).* A function $f(t): Z \to R$ is said to be almost periodic in $t \in Z$ if for any $\varepsilon > 0$, there exists an integer $L_\varepsilon > 0$ such that among any $L_\varepsilon > 0$ consecutive integers there is an integer $p_\varepsilon > 0$ such that

$$sup_{t \in Z} \left| f(t + p_\varepsilon) - f(t) \right| < \varepsilon.$$

Definition 2.2 *(Lenart [25,26]).* A second order time series $\{X_t : t \in Z\}$ is called almost periodically correlated (APC) if both mean $\mu(t) = E(X_t)$ and autocovariance functions $B(t, \tau) = cov(X_t, X_{t+\tau})$, are almost periodic functions at t for every $\tau \in Z$.

As Lenart [25,26], we make the following assumptions:

(A1) *For simplicity, assume that the real-valued time series $\{X_t : t \in Z\}$ is zero-mean.*

(A2) *Assume that the time series $\{X_t : t \in Z\}$ is APC.*

By assumptions (A1) and (A2), the autocovariance function $B(t, \tau)$ has the Fourier representation

$$B(t, \tau) \sim \sum_{\omega \in W_t} a(\omega, \tau) e^{i\omega t},$$

where $a(\omega, \tau)$ are the Fourier coefficients of the form

$$a(\omega, \tau) = \lim_{n \to \infty} \left(\frac{1}{n} \sum_{j=1}^{n} B(j, \tau) e^{-i\omega t} \right),$$

and for fixed τ, the set $W_\tau = \{\omega \in [0, 2\pi) : a(\omega, \tau) \neq 0\}$ is a countable set of frequencies (see Corduneanu [8]; Hurd [22]).

(A3) *Assume that the set $W = \bigcup_{\tau \in Z} W_\tau$, is finite. In otherwise, assume the APC process has spectral mass on lines parallel to the diagonal, $T_j(x) = x \pm \alpha_j$, $j = 1, 2, \ldots, m$, in the two-dimensional spectral plane $[0, 2\pi)^2$.*

Notice that under Assumption (A3), the autocovariance function $B(t, \tau)$ has the Fourier representation

$$B(t, \tau) = \sum_{\omega \in W} a(\omega, \tau) e^{i\omega t}.$$

By Assumptions (A2) and (A3), the spectral measure of the process, defined on the bifrequency plane $[0, 2\pi)^2$, has support contained in the set (see Dehay and Hurd [11])

$$S = \bigcup_{\omega \in W} \left\{ (\nu, \gamma) \in [0, 2\pi)^2 : \gamma = \nu - \omega \right\}.$$

Moreover, the coefficients $a(\omega, \tau)$ are the Fourier transforms of complex measures $r_\omega(\cdot)$, that is,

$$a(\omega, \tau) = \int_0^{2\pi} e^{i\xi\tau} r_\omega(d\xi).$$

The measure r_ω can be identified with the restriction of the spectral measure of the process to the line $\gamma = \nu - \omega$, modulo 2π, where $\omega \in W$.

Note: In this paper, the equalities between frequencies (e.g. $\gamma = \nu - \omega$) are modulo 2π.

(A4) *Assume that the measure r_0 is absolutely continuous with respect to the Lebesgue measure.*