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Brief paper

Absolute stability analysis for negative-imaginary systems ☆

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ABSTRACT

This paper deals with absolute stability of a Lur'e system with positive feedback where the linear subsystem exhibits negative-imaginary frequency response and the nonlinearity connected in feedback is time-invariant, memoryless and slope-restricted. The proposed absolute stability criterion requires the linear subsystem to belong to the *strongly strict negative-imaginary* class. Along with that, positive definiteness of a symmetric matrix needs to be ensured, where the symmetric matrix is obtained by subtracting the dc-gain matrix of the linear subsystem from a strictly positive diagonal matrix with elements indicating the reciprocal of the maximum slope bounds of the nonlinearities. The stability criterion is proved using a *Lur'e–Postnikov*-type Lyapunov function. Numerical examples are presented to demonstrate the proposed results.

1. Introduction

For a Lur'e system (Desoer & Vidyasagar, 1975; Khalil, 2002), the absolute stability (Desoer & Vidyasagar, 1975; Khalil, 2002; Narendra & Taylor, 1973) indicates that the closed-loop has a globally uniformly asymptotically stable equilibrium point at the origin for all nonlinearities in a given sector. In this paper, an absolute stability criterion is searched for a Lur'e system when the linear dynamics belongs to the negative-imaginary (NI) system class (Lanzon & Petersen, 2008; Petersen & Lanzon, 2010) and the nonlinearity connected via positive feedback is time-invariant, memoryless, and slope-restricted.

Negative-imaginary systems have recently been introduced in Lanzon and Petersen (2008) and Petersen and Lanzon (2010) and have readily attracted attention of the system-theoretic research communities. The NI theory has so far been explored and extended in different directions: *viz.*, stability analysis for interconnected NI and strictly NI (SNI) systems (see e.g. Lanzon & Petersen, 2008; Mabrok, Kallapur, Petersen, & Lanzon, 2014a; Petersen & Lanzon, 2010); lossless system properties in Xiong, Petersen, and Lanzon (2012); stability analysis for interconnected systems with 'mixed' properties (see e.g. Das, Pota, & Petersen, 2013; Patra & Lanzon, 2011); controller synthesis and performance analysis (see Song, Lanzon, Patra, & Petersen, 2010, 2012a,b); and in applications with practical relevances (see e.g. Bhikkaji, Moheimani, & Petersen, 2012; Mabrok et al., 2014a; Mabrok, Kallapur, Petersen, & Lanzon, 2014b; Petersen & Lanzon, 2010). But very limited attention has been paid toward finding the absolute stability conditions for NI systems. This serves as the primary motivation behind our present work. In addition, we often encounter practical systems with a linear time-invariant (LTI) part preceded by a saturation, dead-zone or a slope-restricted nonlinearity of similar kind. If these dynamics, in closed-loop, can be modeled as interconnection of NI systems with slope-restricted nonlinearities in positive feedback, then the results of this work could be crucial to ensure asymptotic stability of the overall nonlinear feedback system. Moreover, absolute stability theory can also be readily cast as a robust stability problem (see Haddad & Bernstein, 1993). This indicates the possibilities that the results in this framework might initiate further research in the area of robust control involving NI (or subclass of NI) systems.

Since passivity theorem (Brogliato, Lozano, Maschke, & Egeland, 2007; Khalil, 2002) plays the central role, the interconnection in absolute stability analysis is generally considered to be with negative feedback. The use of *positive feedback in absolute stability framework* makes this present work fundamentally distinct from most of the extensive literature available on absolute stability for slope-restricted nonlinearities (e.g. Haddad, 1997; Haddad &

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Kapila, 1995; Park, 2002; Park, Banjerdpongchai, & Kailath, 1998; Singh, 1984; Suykens, Vandewalle, & De Moor, 1998; Turner, Kerr, & Postlethwaite, 2009; Turner, Kerr, Postlethwaite, & Sofrony, 2010; Zames & Falb, 1968). A part of the work presented in Ouyang and Jayawardhana (2014) is one relevant contribution, where stability of a positive feedback interconnection of a linear system and a counter-clockwise (CCW) Duhem hysteresis operator is analyzed. The analysis is specifically based on the CCW properties (see Angeli, 2006) of the linear system, which are closely related to the NI properties. However, this scheme lacks generality as it is only applicable when the nonlinearity is modeled using a specific approach.

The absolute stability criterion proposed in this work requires the LTI subsystem to satisfy the properties of a subclass of NI systems. This subclass, named strongly strict negative-imaginary (SSNI) systems, is an existing category in the NI literature and is addressed in detail in Lanzon, Song, Patra, and Petersen (2011). Given any multiple decoupled nonlinearity of the time-invariant memoryless slope-restricted family connected to the LTI subsystem via positive feedback, the proposed theorem states that the overall closed-loop system is absolutely stable if the linear dynamics with minimal representation belongs to the strongly strict negative-imaginary system and a dc-gain condition holds. With the help of a sector transformation (see Narendra & Taylor, 1973; Paré, Hassibi, & How, 2001) on the nonlinearity and a subsequent loop-transformation (Paré et al., 2001), the overall closed-loop system takes a decomposed form. Based on the decomposed parts, a *Lur'e–Postnikov*-type Lyapunov function is constructed and global asymptotic convergence of state trajectories to the origin is established for the positive feedback interconnection. The results emerge equally significant from both NI and absolute stability perspectives. Stability conditions in this proposed framework can be easily tested using existing semidefinite programming (SDP) toolboxes.

The rest of the paper is organized as follows: in Section 2 useful notations are given. Section 3 illustrates some fundamental concepts, definitions and lemmas as a background of this work. The classes of subsystems of the feedback interconnection are also defined. The loop-transformation and subsequent decomposition are presented in Section 4 which streamline the main results of this paper. The stability analysis result is discussed in Section 5. Two simple examples are presented in Section 6 which demonstrate the stability result in the proposed framework.

2. Notation

Notation is standard throughout. Let $\mathbb{R}$ and $\mathbb{C}$ be the field of real and complex numbers, respectively and let $\mathbb{R}_+$ correspond to the set of non-negative real numbers. $\mathbb{R}^{m\times n}$ denotes the set of real matrices of dimension $(m\times n)$. $\Re[s]$ indicates the real part of $s\in\mathbb{C}$. The terminology 'proper' transfer function includes both 'strictly-proper' and 'bi-proper' transfer functions. $\mathcal{R}^{m\times n}$ represents the set of all proper real-rational transfer function matrices of dimension $(m\times n)$. $\mathcal{RH}_\infty$ denotes the set of all proper real-rational stable transfer function matrices. Let $(\cdot)^T$ and $(\cdot)^*$ indicate transpose and complex conjugate transpose, respectively. A^{-T} represents the shorthand of $(A^{-1})^T$. The notation $\mathrm{diag}(\mu_1,\mu_2,\ldots,\mu_m)$ is used as a shorthand for the diagonal matrix with diagonal elements $\mu_i,\ i=1,\ldots,m$. Let $\mathcal{L}_2^m[0,\infty)$ indicate the Lebesgue space which consists of all measurable functions $f\ :\ [0,\infty)\ \to\ \mathbb{R}^m$ such that $\int_0^\infty f(t)^T f(t)\,\mathrm{d}t\ <\ \infty$. $\mathcal{L}_{2e}$ denotes the extended Lebesgue space. $G(s)\overset{s}{=}\left[\begin{array}{c|c}A&B\\\hline C&D\end{array}\right]$ indicates a state-space realization of a real-rational proper transfer function matrix $G(s)\in\mathcal{R}^{m\times n}$, that is, $G(s)=C(sI-A)^{-1}B+D$. Similar notation, with $\overset{s}{=}$ replaced by $\underset{min}{\overset{s}{=}}$, is used to denote a minimal realization.

3. Preliminaries

In this section, some useful definitions and lemmas are presented. A brief on the properties of the class of nonlinearities considered in this work is provided and a sector-transformation is discussed, which together streamline the main results of this paper.

Definition 1 (*Brogliato et al., 2007*)**.** A function $f:\mathbb{R}_+\to\mathbb{R}$ is said to be absolutely continuous on $\mathbb{R}_+$ if for all $\epsilon>0$ there exists a $\delta>0$ such that $\sum_{k=1}^{l}|f(b_k)-f(a_k)|<\epsilon$ for every finite number of disjoint intervals (a_k,b_k), $k=1,\ldots,l$, with $[a_k,b_k]\subset\mathbb{R}_+$ and $\sum_{k=1}^{l}(b_k-a_k)<\delta$.

The space of absolutely continuous functions $f:\mathbb{R}_+\to\mathbb{R}$ is denoted by $AC(\mathbb{R}_+)$. If $u(t)\in AC(\mathbb{R}_+)$, then the time derivative $\dot{u}(t):=\frac{\mathrm{d}}{\mathrm{d}t}u(t)$ exists as a measurable function that is bounded almost everywhere.

Definition 2 (*Turner et al., 2010*)**.** A single valued function $\phi:\mathbb{R}\to\mathbb{R}$, differentiable with respect to its input, is said to have a slope-restriction $[0,\mu)$ if

$$0\le\frac{\phi(u)-\phi(v)}{u-v}<\mu\quad\forall u,v\in\mathbb{R},\ u\neq v,\ 0<\mu<\infty\tag{1}$$

holds.

From the lower bound itself, monotonicity of the nonlinearity ϕ is evident. Assuming the nonlinearity is memoryless and $\phi(0)=0$, putting $v=0$ in (1) yields sector-boundedness of ϕ as well with the same bound μ. Let $\phi(\cdot)$ be a nonlinear operator and $\phi\ :\ AC(\mathbb{R}_+)\ \to\ AC(\mathbb{R}_+)$. If $u(t_2)$ is a sufficiently small local perturbation of $u(t_1)$ on an interval $|u(t_1)-u(t_2)|<\delta\ \forall t\in[t_1,t_2]$ with $\delta>0$, then using inequality (1), a local Lipschitz condition on $\phi(\cdot)$ with respect to input u,

$$|\phi(u(t_1))-\phi(u(t_2))|\le\hat{\phi}'\,|u(t_1)-u(t_2)|\,,\tag{2}$$

can be ascertained, where $0\le\phi'(u):=\frac{\mathrm{d}\phi(u)}{\mathrm{d}u}<\hat{\phi}'<\infty$. In this case, $\hat{\phi}'=\mu$ is the *maximum local slope bound* of the nonlinear function. Hereafter, the shorthand notation $\partial\phi\in[0,\mu)$ is used to denote that the function ϕ is slope-restricted, and the finite maximum slope bound is μ. Later in Section 3.2, it will be shown that nonlinearities with slope-restriction of finite sector can be transformed to nonlinearities with slope-restriction of infinite sector.

A brief on the concept of negative-imaginary systems along with relevant definitions and lemmas from the NI literature (see e.g. Ferrante & Ntogramatzidis, 2014; Lanzon & Petersen, 2008; Mabrok, Kallapur, Petersen, & Lanzon, 2011; Petersen & Lanzon, 2010; Xiong, Petersen, & Lanzon, 2010) are presented next. Negative-imaginary systems (say $R(s)$) are Lyapunov stable systems with equal number of inputs and outputs satisfying the frequency domain condition: $j[R(j\omega)-R(j\omega)^*]\ \ge\ 0$ for all $\omega\in(0,\infty)$. In SISO setting, the positive frequency branch of the Nyquist plot of a typical NI system illustrates that the imaginary part of the frequency response is always non-positive; and rather strictly negative for strictly negative-imaginary (SNI) systems. A formal definition of negative-imaginary system is now presented as in Xiong et al. (2010).

Definition 3 (*Xiong et al., 2010*)**.** A square real-rational proper transfer function matrix $R(s)$ is said to be negative-imaginary (NI) if

(1) $R(s)$ has no pole at the origin and in $\Re[s]>0$;

(2) $j[R(j\omega)-R(j\omega)^*]\ \ge\ 0$ for all $\omega\in(0,\infty)$ except values of ω where $j\omega$ is a pole of $R(s)$;