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Brief paper

Energy control of a pendulum with quantized feedback[☆]

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ABSTRACT

The problem of controlling a nonlinear system to an invariant manifold using quantized state feedback is considered by the example of controlling the pendulum's energy. A feedback control law based on the speed gradient algorithm is chosen. The main result consisting in precisely characterizing allowed quantization error bounds and resulting energy deviation bounds is presented.

1. Introduction

Control theory has initially been developed under idealistic assumptions regarding information transmission in a feedback loop. More recently, however, researchers have been increasingly interested in the question of how much information is really needed to perform a desired control task, or conversely, what control objectives can be achieved with a given amount of information. Such considerations arise from applications where scarce communication resources, sensor limitations, or security concerns play a role, and are also motivated by theoretical interest in understanding the interplay between information and control.

Among the various phenomena responsible for a limited amount of information available in a feedback loop, quantization is one of the most basic and widely investigated. By a *quantizer* we mean a function that maps a continuous real-valued system signal into a piecewise constant one taking a finite set of values, thereby encoding this signal using a finite alphabet. Notable early studies of the effect of quantization on the behavior of control systems include (Curry, 1970; Delchamps, 1990; Kalman, 1956; Miller, Mousa, & Michel, 1988), and a brief overview of the recent literature can be found in Sharon and Liberzon (2012).

One approach to analysis of quantized control systems, taken in Liberzon (2003) and elsewhere, involves modeling quantization effects as additive errors. If the controller possesses suitable robustness with respect to such errors, then the system performance can be shown to degrade gracefully due to quantization. In the context of stabilizing an equilibrium, instead of global asymptotic stabilization one typically obtains two nested invariant regions such that all trajectories starting in the larger one converge to the smaller one, a fact usually established by Lyapunov arguments. While robustness to additive errors is automatic for linear systems and linear feedback controllers, for general nonlinear systems the robustness requirements can be quite restrictive and finding a controller meeting such requirements can be challenging (Liberzon, 2003).

Pendulum dynamics is a popular and important benchmark system in control theory. The problem of stabilizing the upright equilibrium, as well as the problem of controlling the pendulum's energy to a desired level, have been widely studied and call for innovative solutions. In particular, it is known (see, e.g., Shiriaev, Egeland, Ludvigsen, & Fradkov, 2001) that the upright equilibrium cannot be globally asymptotically stabilized by continuous feedback. See Angeli (2001), Åström and Furuta (2000), Rantzer and Ceragioli (2001), Shiriaev et al. (2001), Teel (1996) and the references therein for some interesting contributions to pendulum control. More generally, the problem of energy control for Hamiltonian systems was first considered in Fradkov (1996). In Shiriaev and Fradkov (2000, 2001) extended conditions for control of invariant sets were proposed with application to energy control of the pendulum.

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In this paper we consider the problem of controlling the pendulum's energy to a desired level using quantized state feedback. As the nominal feedback law, we choose one based on the *speed gradient method* from Fradkov (1980) (which stabilizes any energy level without quantization). As a candidate Lyapunov function, we choose the squared difference between the current and the desired energy levels (which decreases for the closed-loop system without quantization). We show that in the presence of sufficiently small state quantization errors, even though the Lyapunov function may not always decrease, the time periods on which it may increase and the amount by which it may increase are suitably bounded and decreasing behavior still dominates. Using these properties, we are able to establish that if the initial energy level is not too far from the desired one, then it will remain not too far from it and will eventually become close to it. While this result may appear intuitively not surprising, our main contribution lies in precisely characterizing allowed quantization error bounds and resulting energy deviation bounds.

The rest of the paper is structured as follows. In Section 2 the general problem of the pendulum's energy control using quantized state feedback is described. Our main result is presented in Section 3. Section 4 is devoted to a numerical example demonstrating the performance predicted by the main theorem.

2. Problem formulation

Consider the pendulum equations

$$\ddot{\varphi}(t) = -\frac{g}{l}\sin\varphi(t) + \frac{1}{ml^2}u(t), \qquad (1)$$

where φ is a deviation angle ($\varphi = 0$ at the lower position), u is a controlling torque, g is a gravity acceleration, m and l are the mass and the length of the pendulum respectively.

Assume that $H(\varphi, \dot{\varphi})$ is the full energy of the pendulum, i.e.

$$H(\varphi, \dot{\varphi}) = \frac{1}{2}ml^2\dot{\varphi}^2 + mgl(1 - \cos\varphi).$$

Consider the problem of energy level stabilization of system (1). Let $z = [\varphi, \dot{\varphi}]^T$, $z \in \mathbb{R}^2$. Let h ($h < 2mgl$) be a positive number. Consider a set

$$X_h = \{z : 0 < H(z) \leqslant h\}.$$

Let H_* ($0 < H_* < h$) be desired energy level and the goal function be as follows

$$V(z) = \frac{1}{2}\left(H(z) - H_*\right)^2. \qquad (2)$$

It is required to design a feedback law

$$u = U(z),$$

providing the achievement of the control goal

$$\lim_{t\to\infty} V\left(z(t, z_0)\right) = 0, \qquad (3)$$

where the initial energy level $H(z_0)$ satisfies the following assumption:

$$z_0 \in X_h, \qquad (4)$$

i.e. z_0 belongs to energy layer between 0 and h.

The algorithm design is based on the *speed gradient method* (Fradkov, 1980, 2007; Fradkov & Andrievsky, 2011). According to the speed gradient method it is required to calculate the function $\omega(z, u) = \dot{V}(z)$, i.e. $\omega(z, u)$ is the speed of variation of the quantity V along the trajectories of system (1)

$$\omega(z, u) = (H(z) - H_*)\, B^T z\, u,$$

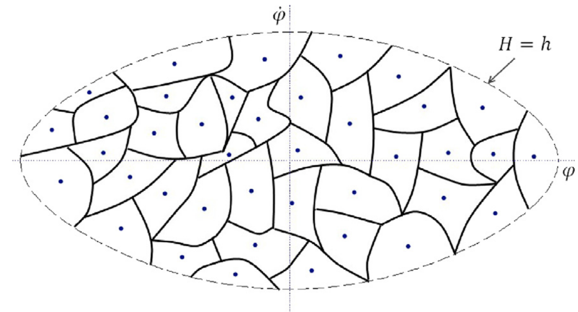

Fig. 1. Quantizer regions.

where $B = [0, \ 1]^T$. Let us find u-derivative of $\omega(z, u)$ and write down the control algorithm in the finite form

$$u = U(z) = -\gamma\,\frac{\partial\omega}{\partial u} = -\gamma\,(H(z) - H_*)\, B^T z, \qquad (5)$$

where $\gamma > 0$.

The idea of algorithm (5) can be explained as follows (Fradkov, 2005). To achieve control goal (3), it is advisable to vary u such that V decreases. But because V does not depend on u, it is difficult to find the direction of such decrease. Instead, one can decrease $\dot{V}$ by ensuring that $\dot{V} < 0$, which is the condition that V decreases. The function $\dot{V}(z) = \omega(z, u)$ explicitly depends on u, which makes it possible to design algorithm (5).

The following theorem, characterizing the performance of control algorithm (5), can be directly concluded from Theorem 3.1 and Remark 3.1 in Fradkov (2007).

Theorem 1. *If the initial energy layer between the levels $H(z_0)$ and H_* does not contain an equilibrium of the unforced system, then the goal level H_* will be achieved in the controlled system (1), (5) for any $\gamma > 0$ from all initial conditions.*

The fulfillment of the condition in Theorem 1 follows from (4).

Let the set $\mathcal{Z} = \{z_i : z_i \in X_h, i \in \mathbb{N}\} \bigcup z_{sat}$ be a finite subset of $X_h \bigcup z_{sat}$, where $z_{sat} \in \mathbb{R}^2$. Consider quantizer $q(z) : \mathbb{R}^2 \to \mathcal{Z}$ proposed in Liberzon (2003). Assume that $Z_i = \left\{z \in \mathbb{R}^2 : q(z) = z_i\right\}$ are quantizer regions (Fig. 1), such that $\bigcup Z_i = X_h$. Hence, $q(z) = z_i$ for all $z \in Z_i$, $i \in \mathbb{N}$. When z does not belong to the union of quantization regions, the quantizer saturates, i.e. $q(z) = z_{sat}$ if $z \notin X_h$.

Suppose that only quantized measurements $q(z)$ of the state z are available. Then the state feedback law (5) is non-implementable. Hence, instead of continuous control (5) consider quantized feedback control law (5):

$$u = U(q(z)) = -\gamma\,(H(q(z)) - H_*)\, B^T q(z), \qquad (6)$$

and control goal

$$\limsup_{t\to\infty} |H(z(t)) - H_*| < \varkappa_1, \qquad (7)$$

where $\varkappa_1$ is some positive number.

Therefore, the problem is to find conditions of achievement of the goal (7) with quantized state feedback control (6). Note that assumption (4) is essential in the case of control algorithm (5) but can be omitted with using modifications of (5). In Shiriaev et al. (2001) it is shown that the global attractivity of the upright equilibrium can be achieved by a modification of the speed gradient energy method based on the idea of variable structure systems (VSS). However, an application of such a modified algorithm to the case of quantized measurements does not seem straightforward and is not pursued here.