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Brief paper

Contraction after small transients[☆]

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ABSTRACT

Contraction theory is a powerful tool for proving asymptotic properties of nonlinear dynamical systems including convergence to an attractor and entrainment to a periodic excitation. We consider three generalizations of contraction with respect to a norm that allow contraction to take place after small transients in time and/or amplitude. These generalized contractive systems (GCSs) are useful for several reasons. First, we show that there exist simple and checkable conditions guaranteeing that a system is a GCS, and demonstrate their usefulness using several models from systems biology. Second, allowing small transients does not destroy the important asymptotic properties of contractive systems like convergence to a unique equilibrium point, if it exists, and entrainment to a periodic excitation. Third, in some cases as we change the parameters in a contractive system it becomes a GCS just before it looses contractivity with respect to a norm. In this respect, generalized contractivity is the analogue of marginal stability in Lyapunov stability theory.

1. Introduction

Differential analysis is based on studying the time evolution of the distance between trajectories emanating from different initial conditions. A dynamical system is called contractive if any two trajectories converge to one other at an exponential rate. This implies many desirable properties including convergence to a unique attractor (if it exists), and entrainment to periodic excitations (Aminzare & Sontag, 2014; Lohmiller & Slotine, 1998; Russo, di Bernardo, & Sontag, 2010). Contraction theory proved to be a powerful tool for analyzing nonlinear dynamical systems, with applications in control theory (Lohmiller & Slotine, 2000), observer design (Bonnabel, Astolfi, & Sepulchre, 2011), synchronization of coupled oscillators (Wang & Slotine, 2005), and more. Recent extensions include: the notion of partial contraction (Slotine, 2003), analyzing networks of interacting agents using contraction theory (Arcak, 2011; Russo, di Bernardo, & Sontag, 2013), a Lyapunov-like characterization of incremental stability (Angeli, 2002), and a LaSalle-type principle for contractive systems (Forni & Sepulchre, 2014). There is also a growing interest in design techniques providing controllers that render control systems contractive or incrementally stable; see, e.g. Zamani, van de Wouw, and Majumdar (2013) and the references therein, and also the incremental ISS condition in Desoer and Haneda (1972).

A contractive system with added diffusion terms or random noise still satisfies certain asymptotic properties (Aminzare & Sontag, 2013; Pham, Tabareau, & Slotine, 2009). In this respect, contraction is a robust property.

In this note, we introduce three forms of generalized contractive systems (GCSs). These are motivated by requiring contraction with respect to a norm to take place only after arbitrarily small transients in time and/or amplitude. Our work was motivated by certain models from systems biology that are not contractive with respect to any (fixed) norm, yet are "almost" contractive. One example is where contraction is lost only on the boundary of the state space, but trajectories emanating from this boundary "immediately" enter the interior of the state space. Thus, we have contraction after an arbitrarily short time transient. The goal of the note is to rigorously define these forms of contraction, study its properties, and derive sufficient conditions for its existence. The contribution

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of this note is thus two-fold: the theoretical study of this type of contraction after an infinitesimal transient, and using this notion to prove important asymptotic properties in applications. Indeed, contraction is usually used to prove *asymptotic* properties, and thus allowing (arbitrarily small) transients seems reasonable. We provide several sufficient conditions for a system to be a GCS. These conditions are checkable, and we demonstrate their usefulness using several examples of systems that are *not* contractive with respect to any norm, yet are GCSs.

In some cases, as we change the parameters in a contractive system it becomes a GCS just before it looses contractivity. In this respect, a GCS is the analogue of marginal stability in Lyapunov stability theory.

We begin with a brief review of some ideas from contraction theory. See Soderlind (2006), Jouffroy (2005) and Rüffer, van de Wouw, and Mueller (2013) for more details, including the historic development of contraction theory, and the relation to other notions.

Consider the time-varying system

$$\dot{x} = f(t, x), \tag{1}$$

with the state x evolving on a positively invariant convex set $\Omega \subseteq \mathbb{R}^n$. We assume that $f(t, x)$ is differentiable with respect to x, and that both $f(t, x)$ and $J(t, x) := \frac{\partial f}{\partial x}(t, x)$ are continuous in (t, x). Let $x(t, t_0, x_0)$ denote the solution of (1) at time $t \geq t_0$ with $x(t_0) = x_0$ (for the sake of simplicity, we assume from here on that $x(t, t_0, x_0)$ exists and is unique for all $t \geq t_0 \geq 0$ and all $x_0 \in \Omega$).

We say that (1) is *contractive* on Ω with respect to a norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ if there exists $c > 0$ such that

$$|x(t_2, t_1, a) - x(t_2, t_1, b)| \leq \exp(-(t_2 - t_1)c)|a - b| \tag{2}$$

for all $t_2 \geq t_1 \geq 0$ and all $a, b \in \Omega$. In other words, any two trajectories contract to one another at an exponential rate. This implies in particular that the initial condition is "quickly forgotten". Note that Lohmiller and Slotine (1998) provide a more general and intrinsic definition, where contraction is with respect to a time- and state-dependent metric $M(t, x)$. Simpson-Porco and Bullo (2014) provide a general treatment of contraction on a Riemannian manifold; see also Lewis (1949). Some of the results below may be stated using this more general framework. But, for a given dynamical system finding such a metric may be difficult; see e.g. Aylward, Parrilo, and Slotine (2008) for an algorithm for finding such contraction metrics using sum-of-squares programming.

Another extension of contraction is incremental stability (Angeli, 2002). Our approach is based on the fact that there exists a simple sufficient condition guaranteeing (2), so generalizing (2) appropriately leads to *checkable* sufficient conditions for a system to be a GCS. Another advantage of our approach is that a GCS retains the important property of entrainment to periodic signals.

Recall that a vector norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ induces a matrix measure $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$ defined by $\mu(A) := \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon}(\|I + \epsilon A\| - 1)$, where $\|\cdot\| : \mathbb{R}^{n\times n} \to \mathbb{R}_+$ is the matrix norm induced by $|\cdot|$. A standard approach for proving (2) is based on bounding some matrix measure of the Jacobian J. Indeed, it is well-known (Russo et al., 2010) that if there exist a vector norm $|\cdot|$ and $c > 0$ such that the induced matrix measure $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$ satisfies $\mu(J(t, x)) \leq -c$, for all $t_2 \geq t_1 \geq 0$ and all $x \in \Omega$ then (2) holds. (This is in fact a particular case of using a Lyapunov–Finsler function to prove contraction Forni & Sepulchre, 2014.)

It is well-known (Vidyasagar, 1978, Ch. 3) that the matrix measure induced by the L_1 vector norm is

$$\mu_1(A) = \max\{c_1(A), \ldots, c_n(A)\}, \tag{3}$$

where

$$c_j(A) := A_{jj} + \sum_{\substack{1\leq i\leq n \\ i\neq j}} |A_{ij}|, \tag{4}$$

i.e., the sum of the entries in column j of A, with non diagonal elements replaced by their absolute values. The matrix measure induced by the L_∞ norm is $\mu_\infty(A) = \max\{d_1(A), \ldots, d_n(A)\}$, where

$$d_j(A) := A_{jj} + \sum_{\substack{1\leq i\leq n \\ i\neq j}} |A_{ji}|, \tag{5}$$

i.e., the sum of the entries in row j of A, with non diagonal elements replaced by their absolute values.

Often it is useful to work with scaled norms. Let $|\cdot|_*$ be some vector norm, and let $\mu_* : \mathbb{R}^{n\times n} \to \mathbb{R}$ denote its induced matrix measure. If $P \in \mathbb{R}^{n\times n}$ is an invertible matrix, and $|\cdot|_{*,P} : \mathbb{R}^n \to \mathbb{R}_+$ is the vector norm defined by $|z|_{*,P} := |Pz|_*$ then the induced matrix measure is $\mu_{*,P}(A) = \mu_*(PAP^{-1})$.

One important implication of contraction is *entrainment* to a periodic excitation. Recall that $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ is called T-*periodic* if $f(t, x) = f(t + T, x)$ for all $t \geq 0$ and all $x \in \Omega$. Note that for the system $\dot{x}(t) = f(u(t), x(t))$, with u an input (or excitation) function, f will be T-periodic if u is a T-periodic function. It is well-known (Lohmiller & Slotine, 1998; Russo et al., 2010) that if (1) is contractive and f is T-periodic then for any $t_1 \geq 0$ there exists a unique periodic solution $\alpha : [t_1, \infty) \to \Omega$ of (1), of period T, and every trajectory converges to α. Entrainment is important in various applications ranging from biological systems (Margaliot, Sontag, & Tuller, 2014; Russo et al., 2010) to the stability of a power grid (Dorfler & Bullo, 2012). Note that for the particular case where f is time-invariant, this implies that if Ω contains an equilibrium point e then it is unique and all trajectories converge to e.

The remainder of this note is organized as follows. Section 2 presents three generalizations of (2). Section 3 details sufficient conditions for their existence, and describes their implications. Due to space limitations, the proofs of all the results are placed at: http://arxiv.org/abs/1506.06613.

2. Definitions of contraction after small transients

We begin by defining three generalizations of (2).

Definition 1. The time-varying system (1) is said to be:

- *contractive after a small overshoot and short transient* (SOST) on Ω w.r.t. a norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ if for each $\varepsilon > 0$ and each $\tau > 0$ there exists $\ell = \ell(\tau, \varepsilon) > 0$ such that
$$|x(t_2 + \tau, t_1, a) - x(t_2 + \tau, t_1, b)| \leq (1 + \varepsilon)\exp(-(t_2 - t_1)\ell)|a - b|$$
for all $t_2 \geq t_1 \geq 0$ and all $a, b \in \Omega$.
- *contractive after a small overshoot* (SO) on Ω w.r.t. a norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ if for each $\varepsilon > 0$ there exists $\ell = \ell(\varepsilon) > 0$ such that
$$|x(t_2, t_1, a) - x(t_2, t_1, b)| \leq (1 + \varepsilon)\exp(-(t_2 - t_1)\ell)|a - b|$$
for all $t_2 \geq t_1 \geq 0$ and all $a, b \in \Omega$.
- *contractive after a short transient* (ST) on Ω w.r.t. a norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ if for each $\tau > 0$ there exists $\ell = \ell(\tau) > 0$ such that
$$|x(t_2 + \tau, t_1, a) - x(t_2 + \tau, t_1, b)| \leq \exp(-(t_2 - t_1)\ell)|a - b| \tag{6}$$
for all $t_2 \geq t_1 \geq 0$ and all $a, b \in \Omega$.

The definition of SOST is motivated by requiring contraction at an exponential rate, but only after an (arbitrarily small) time τ, and with an (arbitrarily small) overshoot $(1 + \varepsilon)$. However, as we will see below when the convergence rate ℓ may depend on ε a somewhat richer behavior may occur. The definition of SO is similar to that of SOST, yet now the convergence rate ℓ depends only on ε,