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Efficient general sparse denoising with non-convex sparse constraint and total variation regularization

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ABSTRACT

In this paper, we proposed an effective and computationally efficient algorithm without iterations, named general sparse denoising with total variation regularization (GSDN-TV), for solving the convex optimization problem of combining the sparse regularization and total variation (TV) regularization. In the GSDN-TV, the original convex optimization problem is divided into two convex optimization subproblems. Each of the subproblems only contains one regularization and can be efficiently solved or has the closed-form solution. The final solution of the original problem can be obtained by solving the two subproblems one by one without iterations. By using the non-convex firm penalty function in the sparse regularization, the GSDN-TV is applied to the wavelet-TV denoising problem and achieves outstanding performances.

1. Introduction

Wavelet denoising with soft-thresholding is an effective and computationally efficient method for noise reduction in the signal processing field. However, the soft-thresholding, which performs the sparsity on the wavelet transform domain, often seems to suffer from some artifacts such as spurious noise spikes and pseudo-Gibbs oscillations [1,2]. To overcome this drawback, an effective way is to use the total variation (TV) denoising by introducing the TV regularization in the spatial/time domain [3–5], but it can induce the undesirable staircase artifacts [6,11].

A successful approach is to combine the sparsity in the wavelet transform domain and TV regularization in spatial or time domain [6,7]. The similar strategy also can be seen in [8–10]. Although these methods can achieve good performance for the signal denoising, an important problem is how to efficiently solve the problem with hybrid regularization [6]. The hybrid regularization can include several regularization terms that do not necessarily act in the same domain (e.g., spatial/time and wavelet transform domains). To solve the optimization problem with hybrid regularization, Pustelnik et al. [6] proposed an accelerated version of the parallel proximal algorithm, and Wang et al. [7] were based on the alternate iterative algorithm by using the variable splitting method. However, all these methods need to make convergence after many iterations.

Recently, Ding and Ivan [11] proposed the unified wavelet-TV (WATV) denoising algorithm based on based on alternating direction method of multipliers (ADMM) algorithm framework. In their method, the sparse regularization in the wavelet-domain is expressed by a non-convex penalty function, because it can induce sparsity more strongly. By choosing a suitable parameter of the non-convex function so that satisfying the convexity condition, the objective function of the denoising problem can remain convex and can be solved based on ADMM algorithm. However, the ADMM algorithm is also not a very effective algorithm, and it introduces extra parameter which needs to be specified by the user [12]. The idea of combining the sparse regularization and TV regularization can be traced back to the problem of the fused lasso [13], in which the problems of sparse denoising and TV denoising are fused together. The fused lasso problem is further extended to the non-convex penalty function as the regularization [14]. Most of these algorithms for solving the fused lasso problem are based on some iterative algorithms, such as majorization-minimization (MM) [12] or ADMM [11].

In this paper, we proposed an effective and computationally efficient algorithm, named general sparse denoising with TV regularization (GSDN-TV), without iterations. In the GSDN-TV, the original convex optimization problem, containing the sparse regularization and TV regularization, is solved by solving two convex subproblems one by one. The GSDN-TV is partly motivated by the works [15,16] and extends them to the more general case. One of the

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subproblems in the GSDN-TV, which only contains the TV regularization, can be efficiently solved by using the TV denoising algorithm [17] by using the taut string algorithm [18]. The other one, which only contains the sparse regularization, can be exactly solved by using the proximity operator of the penalty function of the regularization. We apply the GSDN-TV to the WATV denoising problem by using the firm penalty function [19] in the sparse regularization, because it can induce sparsity more strongly and has the proximity operator in closed-form expression. Moreover, our experimental results show that the proposed WATV algorithm based on the GSDN-TV achieves compatible to the method based on ADMM in [11] but it doesn't need iteration.

Compared with the current algorithms, such as [6,7,11], the main contribution of the paper is twofold. i) We suggest that the sparse denoising problem with non-convex sparse penalty function and TV regularization can be efficiently solved by solving two subproblems one by one without using the iteration algorithms, such as MM and ADMM; ii) We use the non-convex firm penalty function in the sparse regularization to induce sparsity more strongly and give its convexity condition.

2. General sparse denoising with TV regularization

2.1. Problem formulation

The noisy signal $\mathbf{y} \in \mathbb{R}^N$ can be modeled as

$$\mathbf{y} = \mathbf{s} + \mathbf{n} \tag{1}$$

where $\mathbf{s} \in \mathbb{R}^N$ and $\mathbf{n} \in \mathbb{R}^N$ are the clean signal and the noise, respectively. Suppose that $\mathbf{W} \in \mathbf{R}^{N\times N}$ is an orthogonal basis, such as the discrete wavelet basis, satisfying $\mathbf{W}^T\mathbf{W} = \mathbf{I}$. Then, $\mathbf{y}$ and $\mathbf{s}$ can be represented as $\mathbf{y} = \mathbf{Wc}$ and $\mathbf{s} = \mathbf{Wx}$, where $\mathbf{c} \in \mathbb{R}^N$ and $\mathbf{x} \in \mathbb{R}^N$ are their coefficient vectors over the orthogonal basis $\mathbf{W}$. On the other hand, given the signals $\mathbf{y}$ and $\mathbf{s}$, their coefficient vectors can be easily obtained by $\mathbf{c} = \mathbf{W}^T\mathbf{y}$ and $\mathbf{x} = \mathbf{W}^T\mathbf{s}$ based on the orthogonality of $\mathbf{W}$.

Assume that the estimated signal $\mathbf{s}$ is sparse over the orthogonal basis $\mathbf{W}$ and that its first-order difference is also sparse in the signal domain. With these assumptions, the coefficient vector $\mathbf{x}$ of the clean signal $\mathbf{s}$ can be estimated by solving the following optimization problem

$$\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathbb{R}^N}\left\{F(\mathbf{x}) = \frac{1}{2}\|\mathbf{Wx}-\mathbf{y}\|_2^2 + \lambda_1\mathbf{1}^T\phi(\mathbf{x}) + \lambda_2\|\mathbf{DWx}\|_1\right\} \tag{2}$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$ are the regularization parameters and $\mathbf{1}$ represents the vector of all ones. The sparse regularization term $\mathbf{1}^T\phi(\mathbf{x})$ characterizes the sparsity of the coefficient vector $\mathbf{x}$, where $\phi(\mathbf{x})$ denotes the component-wise application of the non-smooth sparsity inducing penalty function $\phi : \mathbb{R} \to \mathbb{R}$, i.e., $[\phi(\mathbf{x})]_i = \phi(x_i)$. The TV regularization term $\|\mathbf{DWx}\|_1$ characterizes the sparsity of the first-order difference of the clean signal $\mathbf{s}$, where the matrix $\mathbf{D} \in \mathbf{R}^{(N-1)\times N}$ denotes the first-order difference matrix, as defined in [11,22].

2.2. General sparse denoising with TV regularization

Consider the following two optimization problems. The first problem is the sparse denoising problem (SDP) and is formulated as

$$\min_{\mathbf{x}\in\mathbb{R}^N}\left\{f_\phi(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}-\mathbf{y}\|_2^2 + \lambda_1\mathbf{1}^T\phi(\mathbf{x})\right\} \tag{3}$$

where $\phi(\mathbf{x})$ denotes the component-wise application of the non-smooth sparsity inducing penalty function $\phi : \mathbb{R} \to \mathbb{R}$, i.e., $[\phi(\mathbf{x})]_i = \phi(x_i)$. Suppose that the objective function $f_\phi(\mathbf{x})$ is convex and then its solution can be uniquely represented by the proximity operator $\theta_\phi : \mathbb{R}^N \to \mathbb{R}^N$ of the penalty function ϕ, that is

$$\theta_\phi(\mathbf{y};\lambda_1) = \arg\min_{\mathbf{x}\in\mathbb{R}^N}\left\{f_\phi(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}-\mathbf{y}\|_2^2 + \lambda_1\mathbf{1}^T\phi(\mathbf{x})\right\} \tag{4}$$

The second problem is the TV denoising problem (TVDP)

$$\min_{\mathbf{x}\in\mathbb{R}^N}\left\{f_{TV}(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}-\mathbf{y}\|_2^2 + \lambda_2\|\mathbf{Dx}\|_1\right\} \tag{5}$$

where $\mathbf{D}$ is the first-order difference matrix. The objective function $f_{TV}(\mathbf{x})$ is convex and it can be solved efficiently, according to [17]. Its solution can be defined by the proximity operator $\theta_{TV} : \mathbb{R}^N \to \mathbb{R}^N$, that is

$$\theta_{TV}(\mathbf{y};\lambda_2) = \mathrm{tvd}(\mathbf{y},\lambda_2) \tag{6}$$

where the $\mathrm{tvd}(\mathbf{y},\lambda_2)$ can be implemented efficiently by the TV denoising algorithm [17] by using the taut string algorithm [18].

With the above SDP and TVDN subproblems, we present the general sparse denoising with TV regularization (GSDN-TV) algorithm in Algorithm 1:

Algorithm 1 General sparse denoising with TV regularization (GSDN-TV) algorithm.

Input:
Noisy signal $\mathbf{y}$ and regularization parameters λ_1, λ_2

Procedure:
(a) Given $\mathbf{y}$, and then solve the TVDP

$$\mathbf{s}_1^* = \theta_{TV}(\mathbf{y},\lambda_2) \tag{7}$$

(b) Let $\mathbf{x}_1^* = \mathbf{W}^T\mathbf{s}_1^*$, and then solve the SDP

$$\mathbf{x}_2^* = \theta_\phi(\mathbf{x}_1^*;\lambda_1) \tag{8}$$

Output:

$$\mathbf{x}^* = \mathbf{x}_2^* \tag{9}$$

Theorem 1. *Assume that the optimization problems in (2) and (3) are convex. Assume that the penalty function $\phi(\mathbf{x})$ in Eq. (3) is permutation invariant, that is, $[\theta_\phi(\mathbf{x})]_m \geq [\theta_\phi(\mathbf{x})]_n$ for $x_m \geq x_n$. The output of the GSDN-TV algorithm in Algorithm 1, $\mathbf{x}^*$, is the solution of the optimization problem (2).*

The proof of Theorem 1 can be found in Appendix A.1.

Note that, the GSDN-TV algorithm extends the results in [15,16]. In their works, the penalty function ϕ of the SDP in (3) for inducing the sparsity is ℓ_1 norm, which is a special case of Algorithm 1 with the regularization $\|\mathbf{x}\|_1$ and the orthogonal basis $\mathbf{W} = \mathbf{I}$.

Theorem 1 states that the optimization problem (2) can be directly solved without using iteration algorithms, such as MM or ADMM, if the penalty function $\phi(\mathbf{x})$ in SDP is permutation invariant and both subproblems can be efficiently solved or have the closed-form solution. The TVDN be effectively solved by using the TV denoising algorithm [17]. The solution of the SDP, however, depends on the selection of the penalty function, i.e., whether the penalty function has a closed-form proximity operator or not.

3. Efficient wavelet-TV denoising algorithm with firm thresholding

3.1. Firm penalty function and convexity condition

Although several non-convex penalty functions (see [22,21] for details), such as logarithmic (log), arctangent (tan), first order ra-

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