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On the numerical generation of positive-axis-defined distributions with an exponential autocorrelation function

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ABSTRACT

Stochastic modeling commonly requires random process generation with an exponential autocorrelation function (ACF). These random processes may be represented as a solution of a stochastic differential equation (SDE) of the first order and usually have one-sided (positive-axis-defined) distributions. However, adoption of the SDE-based method faces serious limitations due to difficulties with the numerical solution. To overcome this issue we propose a tractable general numerical solution of the above-mentioned SDE that preserves solution positivity and accuracy, and validate it with numerical simulations.

1. Introduction

The exponential, or approximately exponential, ACF is amongst the common known signals of natural origin that is also analytically tractable. Hence, the proposed method will be useful in a plethora of fields and applications, including modeling radio and optical wave propagation [1–6], radar [7,8], biology [9], speech processing [10] and more.

Through the years, significant effort has been devoted to the generation of non-Gaussian sequences with an arbitrary probability density function (PDF) and ACF, e.g. [11–14]. While providing useful results, these methods have either limited accuracy, or analytical complexity, or an excessive number of parameters, or limited generated process length. One of the efficient methods for the generation of a process with an arbitrary PDF and an exponential ACF was proposed by Primak et al. [15,16]. The method is based on the solution of properly constructed stationary stochastic differential equations (SDEs) that is derived from a corresponding steady-state Fokker–Planck equation [17–19].

Propagation modeling typically requires the generation of a positive-axis-defined distribution, e.g. Rayleigh, log-normal, Nakagami, Gamma, etc. [20,21]. In principle, a SDE-based method may be effective for the generation of such distributions. However, a non-trivial numerical scheme needs to be applied for solving such SDEs. Moreover, some of these distributions have a discontinuity at the origin, i.e. half-normal distribution and exponential distribution, and, hence, the numerical solution of such SDEs is even more challenging. While preliminary theoretical analysis can be found in [22] and numerical analysis for some special cases can be found in [23], a general and effective numerical solution is still sought.

The numerical solution of a SDE that preserves its positivity is of significant interest at present, due to its importance in mathematical finance theory. In this context, a particular case of the Gamma distribution was rigorously analyzed in [24,25]. Moreover, it was recently shown that such a solution is particularly useful for a special case of SDE, related to the Gamma distribution with an exponential ACF [26]. In this particular case, a numerical solution that preserves positivity was sought by the application of implicit numerical methods. In this paper, we generalize our previous single-distribution results [26] and apply them to the general class of above-mentioned SDEs that are useful for the generation of positive-axis-defined distributions with exponential ACFs. The main contribution of the paper is demonstrating the matching between the theoretical derivation outlined in [15,18] and the numerical integration scheme, whose efficacy is supported herein by numerical examples and statistical validation. In addition, the supplementary Mathematica code is published for the reader's convenience [27].

2. Theory

2.1. Synthesis of SDE

We start with the essential SDE theory. SDEs are the natural extension of ordinary differential equations with the addition of a white noise term, of the form

$$\dot{x} = f(x) + g(x)\xi(t),\ t \geqslant 0 \tag{1}$$

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with some initial condition $x(t_0) = x_0$, where $\xi(t)$ is a normal uncorrelated white Gaussian noise (WGN) process and $f(x)$ and $g(x)$ are time-independent deterministic functions. The time notation was omitted for brevity, $x = x(t)$.

The SDE solution, $x(t)$, which is actually a random process, may be used to model various random phenomena by appropriate selection of functions $f(x)$ and $g(x)$. For the special case, when the desired solution of (1) is an ergodic and wide-sense stationary (WSS) process with an arbitrary distribution, $p_x(x)$, and with an exponential ACF, $C_{xx}(\tau)$, functions $f(x)$ and $g(x)$ are given by [18,28]

$$f(x) = -\lambda\,(x - m_x) \tag{2a}$$

$$g^2(x) = -\frac{2\lambda}{p_x(x)} \int_0^x (s - m_x)\, p_x(s) ds, \tag{2b}$$

where $p_x(x)$ is the desired positive-axis-defined PDF of the form

$$p_x(x) = \begin{cases} p_x(x) & x \geqslant 0 \\ 0 & x < 0 \end{cases}, \tag{3}$$

m_x is the average of $p_x(x)$ and the exponential ACF function is of the form

$$C_{xx}(\tau) = \sigma_x^2 \exp\left(-\lambda\,|\tau|\right), \tag{4}$$

where σ_x^2 is the variance of $p_x(x)$. The substitution of (2a) into (1) results in the considered (Itô form) SDE of the form

$$\dot{x} = -\lambda\,(x - m_x) + g(x)\xi(t). \tag{5}$$

2.2. Numerical solution

The theoretical solution of the SDE is a continuous-time random process. However, when the numerical result is of interest, a discrete-time approximation solution has to be applied. As already mentioned above, the numerical solution has to deal with discontinuity at the origin. In order to preserve numerical stability, the required solution scheme has to be an implicit (backward) one involving both the current process value x_k and the next value x_{k+1}, where indices k and $k+1$ are related to the values at times t_k and t_{k+1}, respectively, such that $\Delta t = t_{k+1} - t_k$ is the process sampling time.

The solution requires a numerical evaluation of the stochastic integral equation (integral form of (1)) of the form

$$x(t) = x(0) + \int_0^t f(x(s))ds + \int_0^t g(x(s))d\xi(s). \tag{6}$$

The important criterion of the solution method is convergence. The strong order of convergence is equal to γ if there exists a constant C such that

$$\mathbb{E}\left[x_n - x(n\Delta t)\right] \leqslant C\,\Delta t^{\gamma}, \tag{7}$$

when Δt is sufficiently small. The common orders (values of γ) are 0.5, 1 and 1.5. The following numerical examples were evaluated with an implicit (backward) Milstein scheme having order 1; solutions for schemes with orders 0.5 and 1.5 will be discussed later. The general scheme is based on a second-order Taylor approximation of (1) and may be described by a discrete-time differential equation of the form [29, Ch. 12, Eq. (2.9)]

$$x_{k+1} = x_k + f(x_{k+1})\Delta t + g(x_k)\sqrt{\Delta t}\xi_k + \frac{1}{2}g(x_k)g'(x_k)\Delta t\left(\xi_k^2 - 1\right), \tag{8}$$

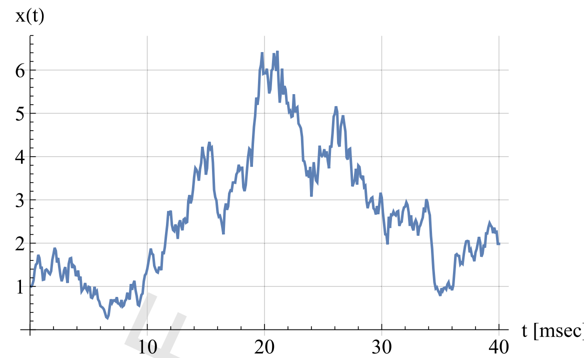

Fig. 1. Samples of the χ^2 distribution with exponential ACF.

where ξ_k are samples of the WGN process. When the numerical solution is applied on (5), the substitution results in a discrete-time differential equation of the form

$$x_{k+1} = \frac{1}{1+\lambda\Delta t}\left[x_k + \lambda m_x \Delta t + g(x_k)\sqrt{\Delta t}\xi_k + \frac{1}{2}g(x_k)g'(x_k)\Delta t(\xi_k^2 - 1)\right]. \tag{9}$$

3. Numerical examples

3.1. χ^2 distribution

The χ^2 distribution is defined by

$$p_x(x;\nu) = \frac{x^{(\nu/2-1)}e^{-x/2}}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)},\; x > 0, \tag{10}$$

where $\Gamma(x)$ is the Gamma function with a mean $m_x = \nu$. The resulting $g^2(x)$ function that is required for the numerical solution of (9) is given by

$$g^2(x_k) = 4x_k\lambda. \tag{11}$$

The simulation results were evaluated for $\nu = 3$, $\lambda = 50$ [1/sec], $\Delta t = 10^{-4}$ [sec] and $k_{max} = 5 \times 10^6$ steps. The resulting (truncated) samples are presented in Fig. 1 and the resulting PDF and ACF are presented in Figs. 2 and 3. These results show a significant resemblance between the theory and the simulation.

In order to validate the statistical relation between the generated process and the given analytical distribution, the Cramér–von Mises, Anderson–Darling, Kolmogorov–Smirnov, Kuiper, Pearson χ^2 and Watson U^2 tests were applied. Since the modification of these tests for correlated data is non-trivial [30], it was applied on samples with $1/\lambda\Delta t$ spacing and revealed that the null hypothesis was not rejected at a remarkably low 0.1% level.

3.2. Half-normal distribution

The PDF of a half-normal distribution is given by

$$p_x(x;\theta) = \frac{2\theta}{\pi}\exp\left(-\frac{x^2\theta^2}{\pi}\right),\; x > 0 \tag{12}$$

with the mean value of $m_x = 1/\theta$. The resulting $g^2(x)$ function that is required for the numerical solution (9) is given by

$$g^2(x_k) == \frac{\lambda\pi}{\theta^2}\left[1 - \exp\left(\frac{x_k^2\theta^2}{\pi}\right)\operatorname{erfc}\left(\frac{x_k\theta}{\sqrt{\pi}}\right)\right], \tag{13}$$

where $\operatorname{erfc}(x)$ is the standard complementary error function. This distribution is of special interest due to its discontinuity at the origin.

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