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Complexity of equilibrium in competitive diffusion games on social networks ☆

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ABSTRACT

In this paper, we consider the competitive diffusion game, and study the existence of its pure-strategy Nash equilibrium when defined over general undirected networks. We first determine the set of pure-strategy Nash equilibria for two special but well-known classes of networks, namely the lattice and the hypercube. Characterizing the utility of the players in terms of graphical distances of their initial seed placements to other nodes in the network, we show that in general networks the decision process on the existence of pure-strategy Nash equilibrium is an NP-hard problem. Following this, we provide some necessary conditions for a given profile to be a Nash equilibrium. Furthermore, we study players' utilities in the competitive diffusion game over Erdos–Renyi random graphs and show that as the size of the network grows, the utilities of the players are highly concentrated around their expectation, and are bounded below by some threshold based on the parameters of the network. Finally, we obtain a lower bound for the maximum social welfare of the game with two players, and study sub-modularity of the players' utilities.

1. Introduction

In recent years, there has been a wide range of studies on the role of social networks in various disciplinary areas. In particular, availability of large data from online social networks has drawn the attention of many researchers to model the behavior of agents in a social network using the possible interactions among them (Bharathi, Kempe, & Salek, 2007; Goyal & Kearns, 2012; Jadbabaie, Lin, & Morse, 2003). One of the widely studied models in social networks is the diffusion model, where the goal is to propagate a certain type of product or behavior in a desired way through the network (Acemoglu, Ozdaglar, & Yildiz, 2011; Goyal & Kearns, 2012; Kempe, Kleinberg, & Tardos, 2003; Young, 2002). Other than applications in online advertising for companies' products, such a model has applications in epidemics and immunization v.s. virus spreading (Khanafer & Başar, 2014; Khanafer, Başar & Gharesifard, 2014). One of the challenges in such models has been obtaining the solution to the best seed placement problem, which has been extensively studied for different processes (Ackerman, Ben-Zwi, & Wolfovitz, 2010; Fazli et al., 2012; Gionis, Terzi, & Tsaparas, 2013; Yildiz, Acemoglu, Ozdaglar, Saberi, & Scaglione, 2011).

In many of the applications in social networks, it is natural to have more than one party that wants to spread information on his own products. This imposes a sort of competition among the providers who are competing for the same set of resources and their goal is to diffuse information on their own product in a desired way across the society. Such a competition can be modeled within a game theoretic framework (Başar & Olsder, 1999), and hence, a natural question one can ask is characterization of the set of equilibria of such a game. Several papers in the literature have in fact addressed this question in different settings, with some representative ones being Bharathi et al. (2007), Brânzei and Larson (2011), Ghaderi and Srikant (2013), Goyal and Kearns (2012), Richardson and Domingos (2002) and Singer (2012). Our goal in this paper is to expand on this literature by addressing the issue of complexity of ascertaining the existence of Nash equilibria for some of these models as well as other models introduced here, as described below.

Due to the complex nature of social events which might be woven with rational decisions, one can find various models aimed at capturing the idea of competition over social networks.

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One of the models that describes such a competitive behavior in networks is known as the competitive diffusion game (Alon, Feldman, Procaccia, & Tennenholtz, 2010). This model can be seen as a competition between two or more parties (types) who initially select a subset of nodes to invest, and the goal for each party is to attract as many social entities to his or her own type as possible. It was shown earlier (Alon et al., 2010) that in general such games do not admit pure-strategy Nash equilibria. It has been shown in Takehara, Hachimori, and Shigeno (2012) that such games may not even have a pure-strategy Nash equilibrium on graphs of small diameter. In fact, the authors in Takehara et al. (2012) have shown that for graphs of diameter 2 and under some additional assumptions on the topology of the network, the diffusion game admits a general potential function and hence an equilibrium. However checking these assumptions at the outset for graphs of diameter at most 2 does not seem to be realistic.

One of the advantages of the diffusion game model is that it captures the simple fact that being closer to player's initial seeds will result in adopting that specific player's type. Moreover, the adoption rule which is involved in the diffusion game is quite simple such that it enables each player to compute its best response quite fast with respect to others (at least for the case of single seed placement), given that all the other players have fixed their actions. On the other hand, as we will see in this paper what makes the analysis of such games more complicated is the behavior of nodes which are equally distanced from the players' seeds. Although there were some recent attempts to characterize these boundary points and show the existence of pure-strategy Nash equilibrium of the diffusion game over different types of networks (Alon et al., 2010; Small & Mason, 2012, 2013), in this work we will address this issue in a more general form, and show that finding an equilibrium for diffusion games is an NP-hard problem over general networks. Therefore, unless P $=$ NP, this strongly suggests that in general the complexity of analyzing such a diffusion game is a hard task despite its simple adoption rule. It requires additional relaxations in the structure of the game in order to make it more tractable. As one possible approach one may consider a probabilistic version of the diffusion game using some techniques from Markov chains or optimization of harmonic influence centrality (Acemoglu, Como, Fagnani, & Ozdaglar, 2013, 2014; Vassio, Fagnani, Frasca, & Ozdaglar, 2014). However, in this work we take a different approach by considering the diffusion game over the well-known Erdos–Renyi random graphs which are commonly used in the literature in order to model social networks.

In a related recent earlier work (Etesami & Başar, 2014), we have characterized the utilities of the players based on the graphical distances of various nodes from the initial seeds. In particular, we have studied the complexity of deciding on the existence of a pure-strategy Nash equilibrium. Here, we characterize the equilibria set for some classes of well-studied networks, and explore some connections between the set of equilibria and the underlying network structure. In particular, we provide some necessary conditions for a given profile of strategies to constitute a Nash equilibrium. Moreover, we consider the diffusion game over Erdos–Renyi graphs and prove some concentration results related to utilities of the players over such networks. Finally we provide a lower bound for the optimal social welfare of the diffusion game over general static networks based on their adjacency matrix.

The paper is organized as follows: in Section 2, we describe the competitive diffusion game and review some of its properties and existing results regarding this model. In Section 3, we determine the set of equilibria of two special but well-studied networks, namely the *lattice* and the *hypercube*. In Section 4, we characterize the utilities of the agents based on the relative locations of the players' initial seed placements, and show that, the decision process on the existence of pure-strategy Nash equilibrium over general undirected networks is an NP-hard problem. In Section 5, we provide two necessary conditions based on the network structure for a given profile to be a Nash equilibrium. In Section 6, we consider the diffusion game model over random graphs and show that asymptotically the utility of the players is highly concentrated around their mean. Furthermore, we provide a lower bound for the expected utility of the players based on the parameters of the random graphs. We end the paper with the concluding remarks of Section 7. Finally, in the Appendix, we provide some complementary results related to sub-modularity as well as lower optimal social welfare of the diffusion game over general fixed networks, which can be used to obtain bounds for the price of anarchy of diffusion games whenever an equilibrium exists.

Notations and conventions: For a positive integer n, we let $[n] := \{1, 2, \ldots, n\}$. For a vector $v \in \mathbb{R}^n$, we let v_i be the ith entry of v. Similarly, for a matrix P, we let P_{ij} be the ijth entry of P and we denote the ith row of P by P_i. We denote the transpose of a matrix P by P'. Moreover, we let I and $\mathbf{1}$ be, respectively, the identity matrix and the column vector of all ones of proper dimensions. Given an integer $k > 0$, we denote the set of all k-tuples of integers by $\mathbb{Z}^k$. For any two vectors $u, v \in Z^k$, we let $u \oplus v$ be their sum vector in mod 2, i.e., $(u \oplus v)_i = u_i + v_i \mod 2$, for all $i = 1, \ldots, k$. We let $\mathcal{G} = (V, \mathcal{E})$ to be an undirected graph with the set of vertices V and the set of edges $\mathcal{E}$. We denote the degree of a vertex x in graph $\mathcal{G}$ by $\mathbf{d}(x)$. Corresponding to $\mathcal{G}$ we let $\mathcal{A}_{\mathcal{G}}$ to be its adjacency matrix, i.e. $\mathcal{A}_{\mathcal{G}}(i, j) = 1$ if and only if $(i, j) \in \mathcal{E}$ and $\mathcal{A}_{\mathcal{G}}(i, j) = 0$, otherwise. Given a graph $\mathcal{G} = (V, \mathcal{E})$ and two vertices $x, y \in V$, we define $d_{\mathcal{G}}(x, y)$ to be the length of the shortest graphical path between x and y. Also, for a set of vertices $S \subseteq V$ and a vertex x, we let $d_{\mathcal{G}}(x, S) = \min_{y \in S}\{d_{\mathcal{G}}(x, y)\}$. For a real number a we let $\lceil a \rceil$ to be the smallest integer greater than or equal to a. We deal in this paper with only pure-strategy Nash equilibrium, and occasionally we will drop the qualifier "pure-strategy".

2. Competitive diffusion game

In this section we introduce the competitive diffusion game as was introduced earlier in Alon et al. (2010) and state some of the existing results for such a model.

2.1. Game model

Following the formulation in Alon et al. (2010), we consider here a network $\mathcal{G}$ of n nodes and two players (types) A and B. Initially at time $t = 0$, each player decides to choose a subset of nodes in the network and place his own seeds. After that, a discrete time diffusion process unfolds among uninfected nodes as follows:

- If at some time step t an uninfected node is neighbor to infected nodes of only one type (A or B), it will adopt that type at the next time step $t + 1$.
- If an uninfected node is connected to nodes of both types at some time step t, it will change to a gray node at the next time step $t + 1$ and does not adopt any type afterward.

This process continues until no new adoption happens. Finally, the utility of each player will be the total number of infected nodes of its own type at the end of the process. Moreover, if both players place their seeds on the same node, that node will change to gray. We want to emphasize the fact that when a node changes to gray, not only will it not adopt any type at the next time step, but also may block the streams of diffusion to other uninfected nodes. We will see later that the existence of gray nodes in the evolution of the process can make any prediction process about the outcome of the diffusion process much more complicated.