Digital Signal Processing 73 (2018) 153–163

Contents lists available at ScienceDirect

Digital Signal Processing

www.elsevier.com/locate/dsp

# Noise benefits parameter estimation in LMMSE sense

Ting Yang [a], Shujun Liu [a,*], Hongqing Liu [b], Mingchun Tang [a], Xiaoheng Tan [a], Xichuan Zhou [a]

[a] College of Communication Engineering, Chongqing University, Chongqing, 400044, China
[b] Chongqing Key Lab of Mobile Communications Technology, Chongqing University of Posts and Telecommunications, Chongqing, 400065, China

ARTICLE INFO

*Article history:*
Available online 23 November 2017

*Keywords:*
Nonlinear system
Linear estimation
Stochastic resonance
Linear minimum mean squared error

ABSTRACT

In this work, the effect of additive noise is studied in order to reduce the mean squared error (MSE) between the input parameter and its linear estimator constructed by the nonlinear system output. To improve the estimation performance, the optimal additive noise that minimizes the MSE of the noise enhanced linear minimum mean squared error (LMMSE) estimator is explored and determined. In addition, in the presence of prior information uncertainty, the estimation performances of the noise enhanced LMMSE are investigated under a constant constraint of the expected value of the output, and the corresponding algorithms are developed to find the optimal additive noise. Finally, two illustrative examples are provided to verify the theoretical results. The performance comparisons conducted between the LMMSE estimator without noise excitation and the optimal noise modified LMMSE estimator demonstrate that noise indeed improves the estimation accuracy under certain conditions.



## 1. Introduction

Noise, viewed as an unwanted signal or a disturbance to the system, always coexists with useful signal [1]. The general consensus is that more noise in the system often leads to less channel capacity and worse detection or estimation performance. Therefore, many algorithms and/or filters are developed to separate the noise from the useful signal in signal processing. However, a counterintuitive phenomenon that noise actually benefits the nonlinear system occurs under certain conditions. The phenomenon of noise enhanced system is called stochastic resonance (SR) that was first proposed by Benzi in 1981 [2], and since then, it has been reported in a variety of research areas [3–11], for example, magnetic systems, optical devices, neural net and electronic circuits, to name a few.

From signal detection theory, the output of certain nonlinear systems can be improved by adding an additive noise to the input or adjusting the background noise level [12–20]. Based on the Neyman–Pearson (NP), Bayesian and Minimax criteria, the noise enhanced detection performance is usually investigated. To measure the improvement obtained in signal detection, several performance indexes are employed, for example, output signal-to-noise ratio (SNR) [3,4], detection probability [14–17], or Bayes risk [18, 19], mutual information (MI) [20].

In addition to the detection, the SR phenomenon has also been observed in estimation problems, where the estimator is constructed utilizing the output of some nonlinear systems [21–25]. The performance of the noise enhanced parameter estimation is usually evaluated in terms of Cramér–Rao lower bound (CRLB) [21, 22] or mean-squared error (MSE). For example, a noise enhanced parameter estimation problem based on quantized observations is studied in [22], which aims to minimize the CRLB for estimating the unknown parameter via adding an additive noise to the observations before the quantization. For a given value of the parameter, the optimal additive noise is proven to be a constant vector. In [23], under the unbiased constraint, a general parameter estimation problem utilizing the additive noise is investigated to minimize the MSE for a deterministic estimator. The conclusion is that in this case the optimal noise is selected as a randomization of no more than two constant vectors. Furthermore, in [24] and [26], the authors demonstrated the possibility of improving the performance of an optimal Bayesian estimator and a quantizer-array linear estimator by operating at higher noise levels. From the discussions above, it is seen that the effects of additive noise and background noise level on MSE have been investigated in [23] for a fixed suboptimal estimator and in [24] for an optimal Bayesian estimator, respectively.

However, the influence of the additive noise that minimizes MSE for an optimal parameter estimator has not been studied yet. The minimum MSE estimator as well as Bayesian estimator is the conditional expectation of the parameter [26], whereas it is hard

* Corresponding author.
*E-mail address:* liusj@cqu.edu.cn (S. Liu).

https://doi.org/10.1016/j.dsp.2017.11.006
1051-2004/

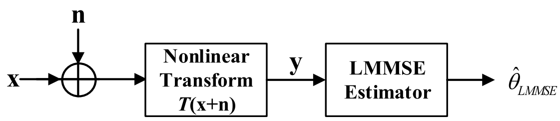


**Fig. 1.** The general noise modified estimation system.

to calculate or may not even be known in real applications since it takes averages against the conditional probability distribution function (pdf) of the parameter. Therefore, the optimal linear estimator is usually considered to avoid this pitfall.

In this paper, the additive noise effect is studied to minimize the MSE between the input parameter and its optimal noise enhanced linear estimator based on the output of nonlinear system. This is the first study on the minimization of the noise modified linear minimum mean-squared error (LMMSE) via adding additive noise to the input of the nonlinear system. To that end, first, a general theory framework of noise modified LMMSE estimation for a nonlinear system is established. Second, the additive noise that minimizes the noise modified LMMSE is explored and proven to be a constant vector. Furthermore, considering the fact of unavoidable uncertainty in the prior information because it is usually estimated by previous experiences in practical applications, the constrained noise enhanced LMMSE optimization problems are studied and the corresponding algorithms are developed. Finally, the estimation performance comparisons of the LMMSE estimators before and after adding noise are conducted to demonstrate the noise effect. In light of this, the main contributions of this paper are summarized as follows:

- Formulation of noise enhanced LMMSE estimator for a nonlinear system is presented.
- Derivations of the optimal additive noise that minimizes noise modified LMMSE are provided.
- Explorations of the minimum noise modified LMMSE under a constant constraint are made.
- Comparisons of the estimation performance before and after adding noise are conducted.

The remainder of this paper is organized as follows. In Section 2, a general framework of noise enhanced LMMSE is formulated for a nonlinear system. The optimal additive noise corresponding to the minimum LMMSE is explored in Section 3. The special case of noise enhanced LMMSE under a constant constraint is discussed in Section 4. Finally, numerical examples are presented in Section 5 to illustrate the theoretical results and the conclusions are made in Section 6.

## 2. Formulation of noise enhanced estimation problem

This study aims to find a suitable additive noise to improve the estimation performance of a nonlinear system. Specifically, we focus on how to reduce the MSE of linear estimation by adding an additive noise to the nonlinear system input. The general noise modified estimation system is shown in Fig. 1.

From Fig. 1, $\mathbf{x} \in \mathbb{R}^N$ represents an $N$-dimension input signal and it is closely related to the unknown parameter $\theta$, whose pdf is denoted by $p_\theta(\theta)$, $\mathbf{n}$ denotes an additive noise with pdf $p_\mathbf{n}(\mathbf{n})$ which is independent from the input signal $\mathbf{x}$, and $\mathbf{y} = T(\mathbf{x}+\mathbf{n})$ is the nonlinear system output, where $T(\cdot)$ represents nonlinear transformation.

In this paper, a linear estimator of the input parameter $\theta$ is obtained based on the nonlinear system output $\mathbf{y}$ in LMMSE sense. The noise modified LMMSE estimator $\hat{\theta}_{LMMSE}(p_\mathbf{n}(\mathbf{n}))$ corresponding to the additive noise with pdf $p_\mathbf{n}(\mathbf{n})$ can be expressed as below [27]:

$$\hat{\theta}_{LMMSE}\big(p_\mathbf{n}(\mathbf{n})\big) = \frac{Cov_{\theta,\mathbf{y}}(p_\mathbf{n}(\mathbf{n}))}{Var_\mathbf{y}(p_\mathbf{n}(\mathbf{n}))} \cdot \mathbf{y} + E(\theta) - \frac{Cov_{\theta,\mathbf{y}}(p_\mathbf{n}(\mathbf{n}))}{Var_\mathbf{y}(p_\mathbf{n}(\mathbf{n}))} E_\mathbf{y}\big(p_\mathbf{n}(\mathbf{n})\big), \tag{1}$$

where $E(\theta) = \int_{\mathbb{R}^N} \theta p_\theta(\theta) d\theta$ denotes the expected value of $\theta$, $Cov_{\theta,\mathbf{y}}(p_\mathbf{n}(\mathbf{n}))$ denotes the covariance of the input parameter $\theta$ and the output $\mathbf{y}$, $E_\mathbf{y}(p_\mathbf{n}(\mathbf{n}))$ and $Var_\mathbf{y}(p_\mathbf{n}(\mathbf{n}))$ represent the expected value and the variance of the system output $\mathbf{y}$, respectively. After simple calculations, the LMMSE between the input parameter $\theta$ and its noise modified LMMSE estimator corresponding to the additive noise with pdf $p_\mathbf{n}(\mathbf{n})$ is

$$LMMSE\big(p_\mathbf{n}(\mathbf{n})\big) = Var(\theta) - \frac{Cov^2_{\theta,\mathbf{y}}(p_\mathbf{n}(\mathbf{n}))}{Var_\mathbf{y}(p_\mathbf{n}(\mathbf{n}))} \tag{2}$$

where $Var(\theta)$ denotes the variance of the input parameter $\theta$, given by

$$Var(\theta) = \int_{\mathbb{R}^N} \big(\theta - E(\theta)\big)^2 p_\theta(\theta) d\theta. \tag{3}$$

In order to obtain $LMMSE(p_\mathbf{n}(\mathbf{n}))$ corresponding to the additive noise with pdf $p_\mathbf{n}(\mathbf{n})$, in (2), the values of $Var(\theta)$, $Cov_{\theta,\mathbf{y}}(p_\mathbf{n}(\mathbf{n}))$ and $Var_\mathbf{y}(p_\mathbf{n}(\mathbf{n}))$ are required. Obviously, $Var(\theta)$ is fixed for any additive noise since $Var(\theta)$ completely depends on the pdf of $\theta$. In addition, $Cov_{\theta,\mathbf{y}}(p_\mathbf{n}(\mathbf{n}))$ is related to both the pdf of $\theta$ and the conditional pdf of the system output $\mathbf{y}$ for a given $\theta$, and $Var_\mathbf{y}(p_\mathbf{n}(\mathbf{n}))$ only hinges on the pdf of the system output $\mathbf{y}$. Since $p_\theta(\theta)$ is assumed to be known, it is necessary to obtain the conditional pdf of the system output $\mathbf{y}$ for a given $\theta$ and the pdf of the system output $\mathbf{y}$ to find the optimal additive noise pdf that minimizes $LMMSE(p_\mathbf{n}(\mathbf{n}))$.

Suppose that the input signal $\mathbf{x}$ is expressed by

$$\mathbf{x} = \varphi(\theta) + \boldsymbol{\nu}, \tag{4}$$

where $\varphi(\theta)$ is any valid function of $\theta$ and $\boldsymbol{\nu}$ denotes the background noise with pdf $p_{\boldsymbol{\nu}}(\boldsymbol{\nu})$. For a given value of $\theta$, the conditional pdf of the system output $\mathbf{y}$ is

$$p_\mathbf{y}(\mathbf{y}|\theta) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \delta\big(\mathbf{y} - T\big(\varphi(\theta) + \boldsymbol{\nu} + \mathbf{n}\big)\big) p_{\boldsymbol{\nu}}(\boldsymbol{\nu}) d\boldsymbol{\nu}\, p_\mathbf{n}(\mathbf{n}) d\mathbf{n}. \tag{5}$$

By integrating out the pdf of $\theta$, the pdf of $\mathbf{y}$ now is expressed as

$$\begin{aligned} p_\mathbf{y}(\mathbf{y}) &= \int_{\mathbb{R}^N} p_\mathbf{y}(\mathbf{y}|\theta) p_\theta(\theta) d\theta \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \delta\big(\mathbf{y} - T\big(\varphi(\theta) + \boldsymbol{\nu} + \mathbf{n}\big)\big) p_{\boldsymbol{\nu}}(\boldsymbol{\nu}) p_\theta(\theta) d\boldsymbol{\nu} d\theta \\ &\quad \times p_\mathbf{n}(\mathbf{n}) d\mathbf{n}. \end{aligned} \tag{6}$$

According to the definitions of $Cov_{\theta,\mathbf{y}}(p_\mathbf{n}(\mathbf{n}))$ and $Var_\mathbf{y}(p_\mathbf{n}(\mathbf{n}))$, the expected value of the system output $\mathbf{y}$ should be calculated first, which is obtained by

$$\begin{aligned} E_\mathbf{y}\big(p_\mathbf{n}(\mathbf{n})\big) &= \int_{\mathbb{R}^N} \mathbf{y} p_\mathbf{y}(\mathbf{y}) d\mathbf{y} \\ &= \int_{\mathbb{R}^N} \mathbf{y} \int_{\mathbb{R}^N} p_\mathbf{y}(\mathbf{y}|\theta) p_\theta(\theta) d\theta d\mathbf{y} \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathbf{y}\delta\big(\mathbf{y} - T\big(\varphi(\theta) + \boldsymbol{\nu} + \mathbf{n}\big)\big) p_{\boldsymbol{\nu}}(\boldsymbol{\nu}) d\boldsymbol{\nu} \end{aligned}$$