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Semi-tensor compressed sensing

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ABSTRACT

In compressed sensing (CS), sparse or compressible signals can be reconstructed with fewer samples than the Nyquist–Shannon theorem requires. Over the past ten years, CS has developed into a relatively mature theory and this brand-new technique has been widely used in many fields such as image processing, wireless communication and medical imaging. In this paper, we propose a new model for signal compression and reconstruction based on semi-tensor product, called STP-CS, which is a generalization of traditional CS. Like traditional CS, we investigate some reconstruction conditions of STP-CS in terms of the spark, the coherence and the restricted isometry property (RIP). The experimental results show that STP-CS has the flexibility to choose a lower-dimensional sensing matrix for signal compression and reconstruction.

1. Introduction

As one of emerging research fields in signal processing, compressed sensing (CS) has attracted considerable attention in recent ten years, because it can reconstruct sparse signals from very few incoherent measurements [1–3]. The standard framework of CS is a special case of underdetermined linear equations

$$\mathbf{y} = \mathbf{\Phi}\boldsymbol{\theta}, \tag{1}$$

where $\mathbf{\Phi}$ is an $m \times n$ matrix with $m < n$. Eq. (1) reflects that the original n-dimensional signal $\boldsymbol{\theta}$ is compressed into an m-dimensional vector $\mathbf{y}$. However, it is impossible to directly reconstruct the original signal $\boldsymbol{\theta}$ from $\mathbf{y}$ because there are infinitely many solutions for Eq. (1). Fortunately, some natural signals can be represented using only a few non-zero coefficients in a suitable basis or dictionary [4,5]. Namely,

$$\boldsymbol{\theta} = \mathbf{\Psi}\mathbf{x}, \tag{2}$$

where $\mathbf{\Psi}$ is a sparsifying dictionary and $\mathbf{x}$ is a sparse vector. We say a vector $\mathbf{x}$ is k-sparse, denoted by $\mathbf{x} \in \sum_k$, if it has at most $k \ll n$ nonzero entries. Thus, we have

$$\mathbf{y} = \mathbf{\Phi}\mathbf{\Psi}\mathbf{x} = \mathbf{A}\mathbf{x}, \tag{3}$$

where $\mathbf{A} = \mathbf{\Phi}\mathbf{\Psi}$ is regarded as the sensing matrix in CS. For a matrix $\mathbf{A}$, the *spark* of $\mathbf{A}$ is the smallest number of columns of $\mathbf{A}$ that are linearly dependent. Donoho and Elad showed that if $spark(\mathbf{A}) > 2k$, then for each measurement vector $\mathbf{y} \in \mathbb{R}^m$ there exists at most one signal $\mathbf{x} \in \sum_k$ such that $\mathbf{y} = \mathbf{A}\mathbf{x}$ [6]. Because finding sparse solutions to underdetermined systems of linear equations is in general NP-hard [1], the sensing matrix $\mathbf{A}$ satisfying such condition is impracticable. Thus, how to construct sensing matrices becomes one of the most important research directions in the field of CS. Fortunately, Candès and Tao proposed a typical criterion for constructing sensing matrix, called restricted isometry property (RIP) [1,7]. A matrix $\mathbf{A}$ satisfies the RIP of order k if there exists a $\delta_k^{\mathbf{A}} \in (0, 1)$ such that

$$(1 - \delta_k^{\mathbf{A}})\|\mathbf{x}\|_2^2 \le \|\mathbf{A}\mathbf{x}\|_2^2 \le (1 + \delta_k^{\mathbf{A}})\|\mathbf{x}\|_2^2 \tag{4}$$

holds for all $\mathbf{x} \in \sum_k$. Random matrices $\mathbf{A} \in \mathbb{R}^{m\times n}$ with entries drawn from a Gaussian distribution, a Bernoulli distribution or more generally a sub-Gaussian distribution have $spark(\mathbf{A}) = m + 1$ with high probability [8]. And such matrices satisfy the RIP with overwhelming probability, providing that $m = O((\delta_k^{\mathbf{A}})^{-2} k \log \frac{n}{k})$ [8,9]. However, such random constructions are often not feasible for real-word applications because some sensing devices with little storage resources are impossible to store all the entries of the sensing matrix when the size of the matrix is very large. To reduce the storage burden, some deterministic approaches for constructing sensing matrices have been proposed, such as structurally subsampled matrices [8], Toeplitz sensing matrices [10], and chaotic sensing matrices [11].

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While the RIP guarantees recovery of k-sparse signals, verifying that a general matrix $\mathbf{A}$ satisfies the RIP has a combinatorial computational complexity, since one must essentially consider $\binom{n}{k}$ submatrices. In many cases it is preferable to use properties of $\mathbf{A}$ that are easily computable to provide more concrete recovery guarantees. The *coherence* of a matrix is one such property. The coherence $\mu(\mathbf{A})$ of a matrix $\mathbf{A}$ is the largest absolute normalized inner product between any two columns of $\mathbf{A}$:

$$\mu(\mathbf{A}) = \max_{1\le i\ne j\le n} \frac{|\langle \mathbf{a}_i, \mathbf{a}_j\rangle|}{\|\mathbf{a}_i\|_2\|\mathbf{a}_j\|_2}, \tag{5}$$

where $\mathbf{a}_i$ denotes the i-th column of $\mathbf{A}$. It is easy to show that $\mu(\mathbf{A}) \in [\sqrt{\frac{n-m}{m(n-1)}}, 1]$ [12,13].

A straightforward approach to obtain the original k-sparse vector $\mathbf{x}$ from Eq. (3) can be viewed as the optimization problem of

$$\mathbf{x}^* = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_0 \text{ subject to } \mathbf{Ax} = \mathbf{y}, \tag{6}$$

which is called l_0 optimization problem. Since the convex property of l_1 norm, a classic method used in compressed sensing is to replace $\|\mathbf{x}\|_0$ with $\|\mathbf{x}\|_1$, i.e.,

$$\mathbf{x}^* = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1 \text{ subject to } \mathbf{Ax} = \mathbf{y}. \tag{7}$$

If $\delta_{2k}^{\mathbf{A}} < \sqrt{2} - 1$, the solution to the l_1 problem is that of the l_0 problem [7]. In addition to l_1-based algorithms, many greedy algorithms, such as orthogonal matching pursuit [14], StOMP [15] and CoSaMP [16], have been proposed for sparse signal reconstruction.

Based on traditional CS, Gan introduced the concept of block compressed sensing (BCS) for natural images [17], where image acquisition is performed in a block-by-block manner through the same sensing matrix. In BCS, an image is first divided into $B \times B$ non-overlapping blocks and then acquired using an appropriately sized sensing matrix. Namely, suppose that $\mathbf{x}_i$ is a vector representing the i-th block of the input image. The corresponding $\mathbf{y}_i$ is then $\mathbf{y}_i = \mathbf{A}_B\mathbf{x}_i$, where $\mathbf{A}_B$ is an $m_B \times B^2$ sensing matrix with $m_B = \lfloor \frac{mB^2}{n} \rfloor$. Because of lightweight reconstruction complexity and lower storage overhead, BCS has been widely used in various multiple-image scenarios, such as video and multi-view imagery [18,19].

In this paper, we propose a new model for signal compression and reconstruction based on semi-tensor product (STP), called STP-CS, which can be viewed as a generalization of tradition CS. This new model breaks the dimension matching condition of the traditional CS model in Eq. (3), i.e., the number of columns of the sensing matrix $\mathbf{A}$ must be equal to the length of the signal $\mathbf{x}$. Under this brand-new model, we first analyze the uniqueness of sparse solution in terms of *spark* and *coherence* from a theoretical point of view. Subsequently, we find that the RIP constant of order k in traditional CS is equal to that in our proposed STP-CS model. It implies that some classical sensing matrices in CS, such as Gaussian, Bernoulli, and Chaotic sensing matrix, can also be used in STP-CS. In addition, we give the exact reconstruction condition on the sensing matrix, which is sufficient for a variety of algorithms to be able to successfully reconstruct the original sparse signal from measurements. At last, the experiment results prove the validity of our theory analysis. Compared to several previous methods for signal compression and reconstruction, the main advantages of our proposed STP-CS can be summarized as follows:

- *Low-storage overhead*. With the help of the semi-tensor product theory, STP-CS can compress high-dimensional signals using lower-dimensional sensing matrices. As a generalization of CS, STP-CS has the flexibility to choose a lower-dimensional sensing matrix. In addition, the experimental results show that the storage overhead of sensing matrices in STP-CS is smaller than that in BCS when the size of each block is not too small.
- *Parallel reconstruction*. The reconstruction algorithm in STP-CS can be implemented in a parallel fashion. The theoretical analysis indicates that a reconstruction instance in STP-CS can be transformed into some independent reconstruction instances in CS. Thus, it can simultaneously perform the reconstruction phase among multiple CS decoders and will lead to the reduction of the total reconstruction time.

The rest of this paper is organized as follows. Section 2 recalls some basic background knowledge of semi-tensor product. We will first introduce the STP-CS model and then give our theoretical results in Section 3. In Section 4, some experiments are carried out to simulate the performance of STP-CS. Section 5 provides a comparison among traditional CS, BCS, and our proposed STP-CS. Last we conclude this paper in Section 6.

2. Semi-tensor product

The concept of STP of matrices was proposed by Cheng et al., which is a generalization of conventional matrix product [20–23]. This novel theory is able to perform matrix multiplication when two matrices do not meet the dimension matching condition. STP has received great attention in a variety of areas, including multilinear algebra [26], game theory [25], and boolean networks [24].

Definition 1. [22] Let $\mathbf{x}$ be a row vector of dimension np, and $\mathbf{y}$ be a column vector with dimension p. Split $\mathbf{x}$ into p equal blocks, named $\mathbf{x}^1, \cdots, \mathbf{x}^p$, which are $1 \times n$ vectors. Define the STP, denoted by $\ltimes$, as

$$\begin{cases} \mathbf{x} \ltimes \mathbf{y} = \sum_{i=1}^{p} \mathbf{x}^i y_i \in \mathbb{R}^{1\times n}; \\ \mathbf{y}^T \ltimes \mathbf{x}^T = \sum_{i=1}^{p} y_i (\mathbf{x}^i)^T \in \mathbb{R}^{n\times 1}. \end{cases} \tag{8}$$

Definition 2. [22] Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{p\times q}$. If either n is a factor of p or p is a factor of n, then we define the STP of $\mathbf{A}$ and $\mathbf{B}$ as the following: $\mathbf{C}$ consists of $m \times q$ blocks as $\mathbf{C} = (\mathbf{c}_{ij})$ and each block is

$$\mathbf{c}_{ij} = \mathbf{a}_i \ltimes \mathbf{b}^j, \ i = 1, 2, \cdots, m, \ j = 1, 2, \cdots, q, \tag{9}$$

where $\mathbf{a}_i$ is the i-th row of $\mathbf{A}$ and $\mathbf{b}^j$ is the j-th column of $\mathbf{B}$.

Given two matrices $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{p\times q}$, the *Kronecker product* between them is defined as

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix} \in \mathbb{R}^{mp\times nq}. \tag{10}$$

Equivalently, we can also define the STP using Kronecker product.

Definition 3. [23] The STP of two matrices $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{p\times q}$ is defined as

$$\mathbf{A} \ltimes \mathbf{B} = (\mathbf{A} \otimes \mathbf{I}_{t/n})(\mathbf{B} \otimes \mathbf{I}_{t/p}), \tag{11}$$

where t is the least common multiple of n and p, i.e., $t = lcm(n, p)$.

Remark 1. Note that $(\mathbf{A} \otimes \mathbf{I}_{t/n}) \in \mathbb{R}^{mt/n\times t}$ and $(\mathbf{B} \otimes \mathbf{I}_{t/p}) \in \mathbb{R}^{t\times qt/p}$, so $\mathbf{A} \ltimes \mathbf{B} \in \mathbb{R}^{mt/n\times qt/p}$.

Remark 2. If $p = n$, then $\mathbf{A} \ltimes \mathbf{B} = (\mathbf{A} \otimes \mathbf{I}_1)(\mathbf{B} \otimes \mathbf{I}_1) = \mathbf{AB}$. It is the standard matrix product.