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Explicit robustness and fragility margins for linear discrete systems with piecewise affine control law[☆]

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ABSTRACT

In this paper, we focus on the robustness and fragility problem for piecewise affine (PWA) control laws for discrete-time linear system dynamics in the presence of parametric uncertainty of the state space model. A generic geometrical approach will be used to obtain robustness/fragility margins with respect to the positive invariance properties. For PWA control laws defined over a bounded region in the state space, it is shown that these margins can be described in terms of polyhedral sets in parameter space. The methodology is further extended to the fragility problem with respect to the partition defining the controller. Finally, several computational aspects are presented regarding the transformation from the theoretical formulations to *explicit* representations (vertex/halfspace representation of polytopes) of these sets.

1. Introduction

When analyzing a control law, both practitioner and theoretician take into account the capacity to cope with disturbances and model uncertainties. This characteristic is classically denoted in control theory as *robustness*. The presence of additive disturbances in the control system structure is due to measurement noises and external perturbation sources. Otherwise, the uncertainty stems from model reduction, linearization of nonlinear elements, imperfect mathematical model or partial information on the parameters. These elements are unavoidable in the control design by the essence of their causes and the practical need of complexity reduction in model-based design, and as a consequence the robustness consideration of the closed-loop is necessary.

This study concentrates on the robustness problem in the presence of model uncertainty for PWA control laws. It is known that in closed loop this class of controllers leads to a hybrid system formulation (Heemels, De Schutter, & Bemporad, 2001). Another motivation for the study of the PWA controllers and their robustness is the recent interest in the optimization-based design via parametric convex programming (Bemporad, Morari, Dua, & Pistikopoulos, 2002; Nguyen, Gutman, Olaru, & Hovd, 2013; Olaru & Dumur, 2004; Seron, Goodwin, & Doná, 2003; Tøndel, Johansen, & Bemporad, 2003) or the approximate explicit solutions in Model Predictive Control (MPC) (Johansen & Grancharova, 2003). Various types of uncertainties exist, in this paper, our interest is in parametric uncertainties, understood as variations of coefficients of a model with a pre-imposed structure. Unstructured uncertainty will generally lead to an augmented state space and the extension of a predefined controller leads to nonuniqueness and related well-posedness problems which are beyond the scope of this study.

At the same time, from the practical point of view, the implementation of control laws in general leads to numerical round-offs. This may affect closed-loop stability. The maximal admissible set of numerical errors, for which the implemented control law still guarantees the stability, is denoted as the *fragility margin*. This problem has already been investigated in literature (Dorato, 1998; Keel & Bhattacharyya, 1997), but these studies neither provide a constructive procedure to compute such a margin, nor cover our interests

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in the class of PWA control laws. As far as it concerns the fragility margin of PWA control laws, we will refer to the possible inaccuracy in the coefficients of the PWA controllers without assuming any uncertainty on the state space partition. Perturbations in the region description will lead to overlapping regions in the partition with implications on non-uniqueness of the trajectories. All these aspects are addressed for the first time in the literature to our best knowledge.

Based on the preliminary results in Olaru, Nguyen, Bitsoris, Rodriguez-Ayerbe, and Hovd (2013); Nguyen, Olaru, Bitsoris, Rodriguez-Ayerbe, and Hovd (2014), this paper provides a theoretical framework and mathematical computation for the explicit robustness/fragility margins of a discrete-time linear system, controlled by a given PWA control law. The methodology is centered around the *robust positive invariance* properties which have been studied since the late '80s (Bitsoris, 1988; Blanchini, 1999; Blanchini & Miani, 2008; Vassilaki, Hennet, & Bitsoris, 1988). Note that the robust positive invariance is associated with robust stability since the trajectories are kept inside a subset of the state space, namely a positively invariant set. Guaranteeing robust asymptotic stability is beyond the scope of this paper. Based on the same constructive principle, the problem of finding the biggest set of errors in the description of the regions of the given state space polyhedral partition is also tackled in this study. The main contribution of this paper is to provide a conceptual advance on the determination of the robustness and fragility margins for a PWA controller and a linear system. Aside from this theoretical aspect, for explicit computations of these margins, computational aspects will also be discussed. These computational aspects rely on vertex/facet enumerations and become expensive once the number of critical regions and dimension increases. However, part of the analysis is independent for each region. Also, all these computations are carried out offline, at the design stage. Therefore, it is reasonable to assume that ample computational power, time and memory are available, making computations of substantial complexity acceptable. This situation is in stark contrast to the online controller computations which typically will be performed under strict real time requirements on low cost computational hardware.

Unlike the robust explicit controllers designs which *a priori* take robustness into account (Kerrigan & Maciejowski, 2004; Kouramas, Panos, Faísca, & Pistikopoulos, 2013; Nguyen, Olaru, & Rodriguez-Ayerbe, 2015), the method presented here allows one to evaluate *a posteriori* the robustness/fragility margins for a given PWA control law. A link can be made between analysis and control design if the fragility/robustness margin is used for retuning PWA controllers to cope with uncertainties while guaranteeing robust positive invariance. However, the robust asymptotic stability should be further elaborated in this case.

Notation and basic definitions

Throughout the paper, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{N}$ and $\mathbb{N}_+$ denote the field of real numbers, the set of nonnegative real numbers, the set of nonnegative integers, the set of positive integer numbers, respectively. For two column vectors: $x, y \in \mathbb{R}^n$, $x = [x_1\ x_2\ \dots\ x_n]^T$, $y = [y_1\ y_2\ \dots\ y_n]^T$, the partial order relation $x \leq y$ is equivalent to $x_i \leq y_i$, $\forall i = 1, \dots, n$. A vector with its elements equal to one (zero) is denoted by $\mathbf{1}$ ($\mathbf{0}$) or by $\mathbf{1}_n$ ($\mathbf{0}_n$) in case the dimension n must be explicitly stated. Similarly, $\mathbf{I}$ denotes an identity matrix of appropriate dimension, with a subscript when the dimension of this matrix needs to be specified i.e. $\mathbf{I}_n$ means $\mathbf{I} \in \mathbb{R}^{n\times n}$. For a matrix $A \in \mathbb{R}^{m\times n}$, then $\mathrm{vec}(A)$ represents the vector composed of the columns of matrix A as follows: $\mathrm{vec}(A) := \begin{bmatrix} A(\cdot,1)^T & \cdots & A(\cdot,n)^T \end{bmatrix}^T$, where $A(\cdot, i)$ denotes the ith column of matrix A. Given two matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, their Kronecker tensor product, denoted by $A \otimes B \in \mathbb{R}^{mp\times nq}$, is defined as:

$$A \otimes B := \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

For an arbitrary set $\mathcal{S} \subseteq \mathbb{R}^n$, $\mathrm{int}(\mathcal{S})$ denotes the interior of $\mathcal{S}$. By $\dim(\mathcal{S})$, we denote the dimension of its affine hull. $\mathcal{V}(\mathcal{S})$ describes the set of vertices whenever $\mathcal{S}$ is a polytope (bounded polyhedral set). If $\mathcal{S} \subset \mathbb{R}^n$ is composed of a finite number of vectors $\mathcal{S} = \{s_1, s_2, \dots, s_m\}$, then $[\mathcal{S}]$ denotes a matrix for which the columns are the elements of $\mathcal{S}$ in an arbitrary order: $[\mathcal{S}] = \begin{bmatrix} s_1 & s_2 & \cdots & s_m \end{bmatrix}$. Moreover, by $\mathrm{conv}(\mathcal{S})$, we denote the convex hull of $\mathcal{S}$. Given a map $f : \mathbb{R}^m \to \mathbb{R}^n$ and a set $\mathcal{S} \subset \mathbb{R}^m$, $f(\mathcal{S}) = \{y \in \mathbb{R}^n \mid \exists x \in \mathcal{S} \text{ such that } y = f(x)\}$ denotes the image of the set $\mathcal{S}$ via the mapping f. For a linear map $f(x) = Ax$ with $A \in \mathbb{R}^{n\times m}$, the image of a set $S \subset \mathbb{R}^m$ is briefly rewritten as $f(\mathcal{S}) = A\mathcal{S}$. The Minkowski sum of two sets P_1 and P_2, denoted as $P_1 \oplus P_2$, is defined as follows:

$$P_1 \oplus P_2 := \{y \mid \exists x_1 \in P_1, x_2 \in P_2 \quad \text{such that } y = x_1 + x_2\}.$$

The unit simplex in $\mathbb{R}^L$ is defined as

$$\mathcal{S}_L = \left\{x \in \mathbb{R}^L_+ \mid \mathbf{1}_L^T x = 1\right\}. \tag{1}$$

Finally, for an $N \in \mathbb{N}_+$, $\mathcal{I}_N$ denotes the set of integers: $\mathcal{I}_N := \{i \in \mathbb{N}_+ \mid i \leq N\}$.

2. Preliminaries

In this section, some basic notions related to the piecewise affine control functions and the discrete dynamics will be introduced to facilitate the problem formulation and the presentation of the main results of the paper.

Definition 2.1. A set of $N \in \mathbb{N}_+$ full-dimensional polyhedra $\mathcal{X}_i \subset \mathbb{R}^n$, i.e. $\mathcal{P}_N(\mathcal{X}) = \{\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_N\}$ is called a *polyhedral partition of a polyhedron* $\mathcal{X} \subseteq \mathbb{R}^n$ if:

(1) $\bigcup_{i\in\mathcal{I}_N} \mathcal{X}_i = \mathcal{X}$.
(2) $\mathrm{int}(\mathcal{X}_i) \bigcap \mathrm{int}(\mathcal{X}_j) = \emptyset$ with $i \neq j$, $(i, j) \in \mathcal{I}_N^2$.

Also, $(\mathcal{X}_i, \mathcal{X}_j)$ are called neighbors if $(i, j) \in \mathcal{I}_N^2$, $i \neq j$ and $\dim(\mathcal{X}_i \cap \mathcal{X}_j) = n - 1$. If $\mathcal{X}$ is a polytope, we call $\mathcal{P}_N(\mathcal{X})$ a *polytopic partition*.

Definition 2.2. A function $f_{pwa} : \mathcal{X} \to \mathbb{R}^m$ defined over a polyhedral partition $\mathcal{P}_N(\mathcal{X})$ of the polyhedron $\mathcal{X}$ by the relation $f_{pwa}(x) = A_i x + a_i$ for $x \in \mathcal{X}_i$, $i \in \mathcal{I}_N$, with $A_i \in \mathbb{R}^{m\times n}$, $a_i \in \mathbb{R}^m$, is said to be a *piecewise affine function* over $\mathcal{P}_N(\mathcal{X})$.

In this paper, we consider discrete linear time-invariant (LTI) systems described by state equations:

$$x_{k+1} = Ax_k + Bu_k, \tag{2}$$

where $x \in \mathbb{R}^n$ represents the state vector, $u \in \mathbb{R}^m$ denotes the control input, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.

If the control action is synthesized in terms of a PWA state feedback defined over a polyhedral partition $\mathcal{P}_N(\mathcal{X})$ of a polyhedron $\mathcal{X} \subseteq \mathbb{R}^n$ then it will be described by

$$u(x_k) = f_{pwa}(x_k) = G_i x_k + g_i \quad \text{for } x_k \in \mathcal{X}_i,\ i \in \mathcal{I}_N, \tag{3}$$

with $G_i \in \mathbb{R}^{m\times n}$ and $g_i \in \mathbb{R}^m$. With this control law, the resulting closed-loop system (2)–(3) is a piecewise affine system described by the state equation:

$$x_{k+1} = (A + BG_i)x_k + Bg_i \quad \text{for } x_k \in \mathcal{X}_i. \tag{4}$$