Automatica 68 (2016) 357–368

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

A linear programming approach to routing control in networks of constrained nonlinear positive systems with concave flow rates ☆

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A R T I C L E I N F O

Article history:
Received 13 March 2014
Received in revised form
31 August 2015
Accepted 20 January 2016
Available online 15 March 2016

Keywords:
Linear programming
Control of networks
Positive systems
Controller constraints and structure

A B S T R A C T

We consider control design for positive compartmental systems in which each compartment's outflow rate is described by a concave function of the amount of material in the compartment. We address the problem of determining the routing of material between compartments to satisfy time-varying state constraints while ensuring that material reaches its intended destination over a finite time horizon. We give sufficient conditions for the existence of a time-varying state-dependent routing strategy which ensures that the closed-loop system satisfies basic network properties of positivity, conservation and interconnection while ensuring that capacity constraints are satisfied, when possible, or adjusted if a solution cannot be found. These conditions are formulated as a linear programming problem. Instances of this linear programming problem can be solved iteratively to generate a solution to the finite horizon routing problem. Results are given for the application of this control design method to an example problem.

1. Introduction

Positive compartmental systems are popular models for describing interconnections of reservoirs whose dynamics are governed by conservation laws and natural positivity and capacity constraints. Examples include automobile or air traffic flows, job-balancing in computer clusters (Fu, Wang, Lu, & Chandra, 2006), or irrigation networks (Cantoni et al., 2007), to name just a few.

A specific class of such compartmental systems, known as Eulerian models, has been used extensively in the air traffic management (ATM) literature (Arneson & Langbort, 2012; Dousse, Arneson, & Langbort, 2012; Le Ny & Balakrishnan, 2009, 2011; Menon, Sweriduk, & Bilimoria, 2004; Menon, Sweriduk, Lam, Diaz, & Bilimoria, 2004; Sun, Strub, & Bayen, 2007; Sun et al., 2006). In an Eulerian model, the aggregate dynamics of groups of aircraft are modeled instead of the dynamics of each individual aircraft. As a result, the order of an Eulerian model depends only on the number of compartments or sections used to describe the network of interest, but not on the total number of vehicles, which greatly reduces complexity in many cases. For more details on Eulerian models and their comparison to Lagrangian models, the reader is referred to Sun et al. (2006).

Previous work in this area (Arneson & Langbort, 2012; Menon, Sweriduk, & Bilimoria, 2004; Menon, Sweriduk, Lam et al., 2004; Sun et al., 2007, 2006) has focused primarily on the use of linear models to describe the outflow of each compartment, or section, of the network. Such models describe the outflow rate of a section as depending linearly on the amount of material in that section.

Here, we focus on concave outflow rate functions. Examples motivating the use of this type of outflow model come from road traffic and air traffic management research. Commonly used models for road traffic rely on a concave flow rate function to describe traffic flow as a function of density. The flow rate increases to a certain point, beyond which, traffic slows, eventually to a stop, as density increases. For example, the cell transmission model for highway traffic introduced in Daganzo (1994) uses a triangular fundamental flow diagram relating density to flow rate. More

☆ This work was funded in part by a NASA Illinois Space Grant Consortium seed grant and NSF grant #0826469 to C. Langbort, and University of Illinois College of Engineering SURGE Fellowship, and NASA Aeronautics Scholarship Program Fellowship to H. Arneson. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Graziano Chesi under the direction of Editor Richard Middleton.

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http://dx.doi.org/10.1016/j.automatica.2016.01.067
0005-1098/Published by Elsevier Ltd.

recently, a triangular outflow rate model was also used in the development of an analytical solution to the traffic flow model in Mazaré, Dehwah, Claudel, and Bayen (2011).

Unlike road traffic, air traffic cannot slow to a stop as aircraft must maintain a minimum speed. The authors of Le Ny and Balakrishnan (2011) point out that, although the outflow of a section of airspace will increase as the density of traffic increases, there is an upper bound on the outflow rate. At low density, flights are allowed to traverse a given section of airspace at their nominal speed and thus more aircraft in a given section results in a greater outflow rate from that section. At low traffic density, it is reasonable to assume a linear relationship between the number of aircraft in the section and the outflow rate of that section. However, aircraft must maintain a minimum separation for safety considerations, and thus, as traffic density increases, separation requirements cause the outflow rate to saturate. A nonlinear, saturating outflow model is proposed in Le Ny and Balakrishnan (2011) and is shown to more accurately capture this saturating effect in dense traffic problems. In that work, the authors generate an increasing, concave, saturating outflow rate curve empirically through a discrete-event simulation which includes separation requirements.

Air traffic flow management is currently performed by human air traffic controllers. Traffic management coordinators (TMCs) react to flow constraints in the National Airspace System (NAS) by implementing air traffic management initiatives in order to safely route air traffic through the NAS. Decisions are made based on TMCs' previous experience with similar conditions affecting a specific region of airspace. Each TMC is likely to react differently given identical conditions. Unfortunately, decisions made in this way are often overly conservative and do not include guarantees on performance. Air traffic is expected to increase steadily over the next two decades (Price, 2015). Air traffic controllers will increasingly rely on automation of this process in order to keep up with the growing demand.

Our previous work Arneson and Langbort (2012) and Dousse et al. (2012) has focused on variations of the problem of routing design for positive compartmental systems to satisfy time-varying capacity constraints motivated by problems in air traffic management. Airspace sector capacity specifies the maximum number of aircraft that a trained human air traffic controller can safely route through the sector. Sector capacity depends on controller workload associated with traffic flow patterns in the sector. Additionally, the number of aircraft that can safely be routed through a sector of airspace at a given time can depend on the weather conditions in the sector at that time. Work has been done to estimate and predict sector capacities (Song, Wanke, & Greenbaum, 2006) in the presence of severe weather conditions.

In Arneson and Langbort (2012), we presented a solution to this problem for single destination networks with linear section outflow rates. The extension of this control design technique to the multiple destination problem is straightforward. Given linear outflow rates, a separate, decoupled network can be created for each destination. Routing solutions can be found individually for each sub-network using the technique presented in Arneson and Langbort (2012).

We focused on single destination networks with nonlinear section outflow rates in Dousse et al. (2012). In contrast to the linear outflow problem of Arneson and Langbort (2012), extending this technique to the multiple destination problem is not straightforward because, with nonlinear outflow rates, sub-networks for each destination exhibit nonlinear coupling. This nonlinear coupling makes the derivation of routing solutions more challenging.

Here, we extend our former work and present a solution to the problem for a multiple destination network with nonlinear outflow rates. Rather than formulating constraints, and subsequently a linear programming (LP) problem, to solve this problem directly, we make use of the solution for the multiple destination network with linear outflow rates. Additional constraints are imposed on the routing solution for the latter so that the resulting closed-loop system with linear outflow rates behaves like the system with nonlinear section outflow rates. These constraints are nonlinear in control design variables and thus cannot be incorporated into the LP problem. To address this issue, we treat these constraint values as fixed and iteratively solve instances of the LP problem, adjusting the fixed values at each iteration.

While our motivation is the application of this control design technique to problems in air traffic management for which saturating outflow rates arise naturally, the control technique is applicable to any concave outflow rate function. We thus derive the control design technique for the more general concave outflow rate function with the saturating outflow rate function being a special case.

The multiple destination network with nonlinear outflow rates, control design objectives which ensure that basic network properties hold, and the formal problem statement are presented in Section 2. The derivation of the control design technique is presented in several stages in Section 3. First, the extension of the control design technique of Arneson and Langbort (2012) for the multiple destination network with linear section outflow rates, formulated as an LP problem, is described in Section 3.1. Control design for the multiple destination network with nonlinear outflow rates is then developed in Section 3.2. Constraints on the control input which force the multiple destination linear outflow rate system to behave like the multiple destination nonlinear outflow rate system are developed and incorporated as additional constraints in the LP problem of Section 3.1. A method to recover a routing strategy for the nonlinear outflow rate system from a solution to this modified LP problem is given. Finally, an algorithm is presented to iteratively solve instances of this modified LP problem. In Section 4, an application example is given to demonstrate the proposed solution method.

Notation. We are often concerned with vectors, scalars and elements of matrices which are indexed over some range. Therefore, for every positive integer N, we define $[N] = \{1, 2, \ldots, N\}$. The cone of entry-wise non-negative vectors of dimension N is denoted by $\mathbb{R}_+^N$. We write $\boldsymbol{x} \geq 0$ to mean that $\boldsymbol{x} \in \mathbb{R}_+^N$, and $\boldsymbol{x} > 0$ if it is in its interior.

2. Problem description

2.1. Model

We consider positive conservative systems, which can be used to describe the flow of material through a network of interconnected sections. Sections may represent, for example, reservoirs in an irrigation network, portions of the road in a traffic model, or volumes of air space in an air traffic flow management problem. Material flows between these sections making its way from one of several sources to a particular sink.

We begin by describing the dynamics of an N-section network with R distinct destinations or sinks. In addition to satisfying any performance objective, the routing solution must also ensure that all material reaches its intended destination. We aggregate material within each section of the network based on final destination. That is, assuming that there are R distinct destinations, we create R coupled sub-networks. Each sub-network describes the flow of traffic through the N-section network for a particular destination $r \in [R]$.

Some of the sections, referred to as "final sections", are sinks through which material exits the network. The set of final sections