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Brief paper

Minimal eventually positive realizations of externally positive systems ☆

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ABSTRACT

It is a well-known fact that externally positive linear systems may fail to have a minimal positive realization. In order to investigate these cases, we introduce the notion of minimal eventually positive realization, for which the state update matrix becomes positive after a certain power. Eventually positive realizations capture the idea that in the impulse response of an externally positive system the state of a minimal realization may fail to be positive, but only transiently. As a consequence, we show that in discrete-time it is possible to use downsampling to obtain minimal positive realizations matching decimated sequences of Markov coefficients of the impulse response. In continuous-time, instead, if the sampling time is chosen sufficiently long, a minimal eventually positive realization leads always to a sampled realization which is minimal and positive.

1. Introduction

The positive realization problem for an externally (i.e., input–output) positive linear system consists in finding a state space representation which is itself positive, i.e., a triple $\{A, b, c\}$ for which A, b and c are nonnegative. The problem has been investigated for several decades, see Benvenuti and Farina (2004) and Farina and Rinaldi (2000) for an introduction and a survey of the main results.

Unlike existence, which is well-characterized (Anderson, Deistler, Farina, & Benvenuti, 1996; Farina & Benvenuti, 1995; Maeda & Kodama, 1981; Ohta, Maeda, & Kodama, 1984), the problem of constructing positive realizations of minimal order is a difficult one, far from being completely solved. The positivity constraints, in fact, imply that not all externally positive systems admit a realization which is both minimal and positive, i.e., which is simultaneously positive, controllable and observable. There is by now a consistent literature on the topic, notably dealing with conditions on the order of the achievable realizations (Hadjicostis, 1999; Nagy & Matolcsi, 2003), and developing algorithms to construct positive realizations in special cases (Benvenuti, 2013; Benvenuti, Farina, Anderson, & De Bruyne, 2000; Canto, Ricarte, & Urbano, 2007; Kim, 2012; Nagy & Matolcsi, 2005). However, systematic procedures for obtaining minimal positive realizations are in general unknown.

Rather than contributing to this search, the scope of this paper is to investigate the structure of the minimal realization of externally positive systems, and to suggest a class of minimal realizations capturing the gap between external and internal positivity. The starting point of our approach is the observation that the fundamental mathematical principle behind positivity is the Perron–Frobenius theorem. In essence, existence of a positive realization is associated to existence of a polyhedral cone which is A-invariant (Anderson et al., 1996; Ohta et al., 1984). Such cone contains the positive eigenvector associated to the dominant eigenvalue, and at least when dominance is strict and the input is vanishing, the evolution of the linear system tends to become aligned with that eigenspace. If we relax the assumption of positivity of A while maintaining the condition that the eigenvector must be contained in $\mathbb{R}^n_+$, then we still have that the free evolution of the state of a minimal realization becomes positive after a transient. Matrices A having both left and right dominant eigenvector in $\mathbb{R}^n_+$ form a special class of matrices called eventually positive, see Altafini and Lini (2015) and Noutsos (2006). While these matrices can have negative entries (hence they do not correspond to positive realizations), they have the property that after a certain power they become positive matrices. Therefore in discrete-time this property naturally leads to free evolutions of the state variables that become nonnegative after a certain

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number of steps. If in addition the reachable cone associated to the realization is contained in $\mathbb{R}^n_+$, then the entire state vector must become positive after a transient. Notice that our approach is qualitatively different from Guidorzi (2014), where relaxing the positivity of A may lead to state trajectories which never become positive when initialized outside the reachable cone.

Formalizing our argument, we show in the paper that a minimal realization in which A is eventually positive and b (resp. c) belongs to the corresponding A-invariant cone (resp. dual cone) is guaranteed to be externally positive. As for the converse, we provide constructive procedures to obtain a minimal eventually positive realization from a given externally positive system. While we do not have an explicit proof that every externally positive system admits a minimal realization of this type, it is tempting to conjecture that indeed it is so, at least in the case of a simple strictly dominant pole.

With respect to a preliminary version of this manuscript appearing as a conference paper (Altafini, 2015), the constructive procedure proposed here (Algorithm 1) is more general, and recovers the result of Altafini (2015) as a special case (Algorithm 2). Such special case is instrumental to show that a consequence of the existence of a minimal eventually positive realization is that the sequence of Markov parameters that compose the impulse response has decimated subsequences for which a minimal positive realization exists, and can be found downsampling the eventually positive realization. The number of steps it takes for A to become positive gives a lower bound on the sought decimation factor. In continuous-time, instead, provided the sampling time is chosen sufficiently high, the sampled system obtained from a minimal eventually positive realization is itself positive and minimal. Also in this case (which is not treated in Altafini, 2015), once an eventually positive realization is available, a lower bound on the sampling time leading to a minimal positive sampled realization is known. These results on sampled/downsampled systems can be interpreted as a dual of the usual Nyquist–Shannon theorem: instead of seeking for a sampling frequency sufficiently high so as to preserve all interesting frequencies of the system, if one selects a sampling frequency enough low it is possible to achieve an internal minimal representation of the system which remains positive, because it disregards the high frequency content. In externally positive systems with strict dominance of the real eigenvalue, as we consider here, these frequencies are associated to non-dominant modes, hence they are necessarily transient. Therefore the nonpositive entries of our eventually positive realizations and the violations of positivity in the state vectors they induce must necessarily be associated to the non-dominant modes.

2. Linear algebra background

For a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, in this paper $A \geq 0$ means $a_{ij} \geq 0$ for any $i, j \in 1, \dots, n$, and $A \neq 0$, while $A > 0$ means $a_{ij} > 0$ for all $i, j = 1, \dots, n$. The matrix A is *nonnegative* (resp. *positive*) if $A \geq 0$ (resp. $A > 0$). This notation is used also for vectors. The spectrum of A is denoted $\mathrm{sp}(A) = \{\lambda_1(A), \dots, \lambda_n(A)\}$, where $\lambda_i(A)$, $i = 1, \dots, n$, are the eigenvalues of A. The *spectral radius* of A, $\rho(A)$, is the smallest real positive number such that $\rho(A) \geq |\lambda_i(A)|$, $\forall i = 1, \dots, n$.

2.1. Eventually positive matrices

Definition 1. A matrix $A \in \mathbb{R}^{n\times n}$ has the *strong Perron–Frobenius property* if $\rho(A)$ is a simple positive eigenvalue of A s.t. $\rho(A) > |\lambda|$ for every $\lambda \in \mathrm{sp}(A)$, $\lambda \neq \rho(A)$, and v, the right eigenvector relative to $\rho(A)$, is positive.

Denote $\mathscr{PF}_n$ the set of matrices in $\mathbb{R}^{n\times n}$ that possess the strong Perron–Frobenius property. These properties are associated to a class of matrices called *eventually positive* (Elhashash & Szyld, 2008; Friedland, 1978; Johnson & Tarazaga, 2004; Noutsos, 2006), class that is strictly bigger than that of positive matrices, in the sense that the matrices can contain negative entries.

Definition 2. A real square matrix A is said to be *eventually positive* if $\exists\, \eta_o \in \mathbb{N}$ such that $A^\eta > 0$ for all $\eta \geq \eta_o$.

The smallest integer η_o of Definition 2 is called the power index of A. Following Olesky, Tsatsomeros, and van den Driessche (2009), eventually positive matrices will be denoted $A \overset{\vee}{>} 0$. For eventually positive matrices a necessary and sufficient condition for the fulfillment of the strong Perron–Frobenius property is available.

Theorem 1 (*Noutsos, 2006, Theorem 2.2*)**.** *For $A \in \mathbb{R}^{n\times n}$ the following are equivalent:*

(1) *Both A, $A^T \in \mathscr{PF}_n$;*
(2) *$A \overset{\vee}{>} 0$;*
(3) *$A^T \overset{\vee}{>} 0$.*

A matrix A is said *exponentially positive* if $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} > 0\, \forall\, t \geq 0$, and A is exponentially positive if and only if A is *Metzler*, i.e., $a_{ij} \geq 0\, \forall\, i \neq j$ (Noutsos & Tsatsomeros, 2008).

Definition 3. A matrix $A \in \mathbb{R}^{n\times n}$ is said *eventually exponentially positive* if $\exists\, t_o \in [0, \infty)$ such that $e^{At} > 0\, \forall\, t \geq t_o$.

We denote the smallest such t_o the exponential index of A. The relationship between eventual positivity and eventual exponential positivity is given in Noutsos and Tsatsomeros (2008).

Theorem 2 (*Noutsos & Tsatsomeros, 2008, Theorem 3.3*)**.** *Given $A \in \mathbb{R}^{n\times n}$, A is eventually exponentially positive if and only if $\exists\, d \geq 0$ such that $A + dI \overset{\vee}{>} 0$.*

2.2. Invariant cones and eventually positive matrices

A set $\mathscr{K} \subset \mathbb{R}^n$ is called a convex cone if $\alpha_1 x_1 + \alpha_2 x_2 \in \mathscr{K}$ $\forall x_1, x_2 \in \mathscr{K}, \alpha_1, \alpha_2 \geq 0$. $\mathscr{K}$ is called solid if the interior of $\mathscr{K}$, $\mathrm{int}(\mathscr{K})$, is nonempty, and pointed if $\mathscr{K} \cap (-\mathscr{K}) = \{0\}$. A proper cone is a closed, pointed, solid cone. A cone is polyhedral if it can be expressed as the nonnegative combination of a finite number of generating vectors $\omega_1, \dots, \omega_\mu \in \mathbb{R}^n$:

$$\mathscr{K} = \mathrm{cone}(\Omega) = \left\{ x = \Omega\alpha = \sum_{i=1}^{\mu} \alpha_i \omega_i, \alpha_i \geq 0 \right\}, \quad (1)$$

where $\Omega = \begin{bmatrix}\omega_1 \dots \omega_\mu\end{bmatrix} \in \mathbb{R}^{n\times\mu}$, $\alpha = \begin{bmatrix}\alpha_1 \dots \alpha_\mu\end{bmatrix}^T \in \mathbb{R}^\mu_+$. It is well-known that alternatively to the "vertices description" (1) for $\mathscr{K}$ one can use the "face description"

$$\mathscr{K} = \{x \text{ s.t. } \Gamma x \geq 0\}, \quad \Gamma \in \mathbb{R}^{\mu\times n}.$$

The pair $\{\Omega, \Gamma\}$ forms a "double description pair" for $\mathscr{K}$. Let $\mathscr{K}^* = \{y \in \mathbb{R}^n \text{ s.t. } y^T x \geq 0\ \forall\, x \in \mathscr{K}\}$ be the dual cone of $\mathscr{K}$. In terms of the double description pair $\{\Omega, \Gamma\}$, we have:

$$\mathscr{K}^* = \{y \text{ s.t. } y = \Gamma^T \beta,\ \beta \geq 0\} = \{y \text{ s.t. } \Omega^T y \geq 0\},$$

i.e., $\{\Gamma^T, \Omega^T\}$ is a double description pair for $\mathscr{K}^*$.

Given $A \in \mathbb{R}^{n\times n}$, the cone $\mathscr{K}$ is said A-invariant if $A\mathscr{K} \subseteq \mathscr{K}$. For an A-invariant cone $\mathscr{K}$, A is said $\mathscr{K}$-positive if $A(\mathscr{K} \setminus \{0\}) \subseteq \mathrm{int}(\mathscr{K})$, i.e., A maps any nonzero element of $\mathscr{K}$ into $\mathrm{int}(\mathscr{K})$. Notice that if A is $\mathscr{K}$-positive then A is $\mathscr{K}$-irreducible, i.e., it does not leave any of the faces of $\mathscr{K}$ invariant (except for $\{0\}$ and $\mathscr{K}$ itself).