Automatica 68 (2016) 162–168

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Approximation of linear distributed parameter systems by delay systems ☆

Michaël Di Loreto, Sérine Damak, Damien Eberard, Xavier Brun

Université de Lyon, Laboratoire Ampère, INSA-Lyon, 20 Avenue Albert Einstein, Villeurbanne 69621, France

ARTICLE INFO

Article history:
Received 6 March 2015
Received in revised form
11 December 2015
Accepted 15 January 2016
Available online 22 February 2016

Keywords:
Distributed parameter systems
Time-delay systems
Approximation
Stabilization

ABSTRACT

The present work addresses continuous-time approximation of distributed parameter systems governed by linear one-dimensional partial differential equations. While approximation is usually realized by lumped systems, that is finite dimensional systems, we propose to approximate the plant by a time-delay system. Within the graph topology, we prove that, if the plant admits a coprime factorization in the algebra of BIBO-stable systems, any linear distributed parameter plant can be approximated by a time-delay system, governed by coupled differential–difference equations. Considerations on stabilization and state-space realization are carried out. A numerical method for constructive approximation is also proposed and illustrated.

1. Introduction

In the input–output approach, approximating a plant means to find a more tractable model with respect to given specifications for analysis, simulation and control synthesis. For distributed parameter plants governed by linear 1-D partial differential equations (Curtain & Morris, 2009; Curtain & Zwart, 1995; Foias, Özbay, & Tannenbaum, 1996; Zwart, 2004), approximation is, in general, realized by lumped models described by linear ordinary differential equations (Myint-U & Debnath, 2007). In feedback and interconnection perspectives, the graph topology of input–output systems, introduced in Vidyasagar (1984, 1985) to handle a topology where feedback is a robust property, gives a supporting structure for approximation. See also Bonnet and Partington (1999), Mäkilä and Partington (1993), Partington (1991), Zhu (1988) for generalizations and metrics of the graph topology for distributed parameter plants. Approximation by lumped models in the graph topology was formalized in Vidyasagar and Anderson (1989), and studied further in Morris (1994, 2001), Partington and Glover (1990) for various metrics. A preliminary result in lumped stabilization for time-delay systems, viewed as a subclass of distributed parameter plants, was also obtained in Kamen, Khargonekar, and Tannenbaum (1985). These methods are mainly based on algebraic representations of input–output plants, and in particular on coprime factorizations (see Logemann, 1993, Loiseau, 2000, Quadrat, 2003 and Vidyasagar, 1985 for time-delay systems). In Vidyasagar and Anderson (1989), existence of a lumped approximation in the graph topology is studied, and gives a clear interpretation of purely atomic parts in coprime factorizations that cannot be closed to lumped distributions, in the BIBO-stability framework. Instead of relaxing graph topology, we propose in this paper to generalize the approximation class of lumped systems by time-delay systems. While this class is still of infinite dimension and can exactly represent some partial differential equations (Răsvan, 2009), state-space realization and advanced tools for analysis and control reach nowadays maturity, and can therefore be of benefit to the study of distributed parameter systems. The main contribution of this paper is to show that, from a quantitative point of view, such a time-delay approximation always exists if a coprime factorization in the Wiener algebra of bounded input bounded output (BIBO) plants exists. This condition appears to be of practical relevance in constructive feedback perspective (Vidyasagar, Schneider & Francis, 1982).

The paper is organized as follows. In Section 2, we briefly recall definitions and introduce some notations. Approximation by time-delay systems is developed in Section 3. In Section 4, we discuss on the state-space realization of the approximation class, and propose a numerical method with explicit construction. Two examples are illustrated in Section 5.

☆ This work was supported by ANR project AIDS from the French National Research Agency, under grant ANR-11-JS03-0004. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Hitay Ozbay under the direction of Editor Miroslav Krstic.

E-mail addresses: michael.di-loreto@insa-lyon.fr (M. Di Loreto), serine.damak@insa-lyon.fr (S. Damak), damien.eberard@insa-lyon.fr (D. Eberard), xavier.brun@insa-lyon.fr (X. Brun).

http://dx.doi.org/10.1016/j.automatica.2016.01.065

2. Preliminaries

In this section, some basic background and notations, that will be used throughout the paper, are introduced.

2.1. Input–output representation

A causal input–output convolution system is a dynamical system described by an equation of the form

$$y(t) = (Pu)(t) \doteq \int_0^t P(\tau)u(t-\tau)\,\mathrm{d}\tau \tag{1}$$

where $y(\cdot)$, $u(\cdot)$ and $P(\cdot)$ are said to be the output, input and kernel of the plant, respectively. For our purpose, we consider the Wiener algebra of BIBO-stable systems (Callier & Desoer, 1978), and we denote it by $\mathscr{A}$. We say that $P \in \mathscr{A}$ if $P = P_{\mathrm{na}} + P_{\mathrm{pa}}$ for $t \geq 0$ and 0 elsewhere, where $P_{\mathrm{na}}(\cdot) \in L_1(0,\infty)$ (non atomic part), that is P_{na} is a real-valued function such that $\|P_{\mathrm{na}}\|_{L_1} = \int_0^\infty |P_{\mathrm{na}}(t)|\,\mathrm{d}t$ is finite. The real-valued distribution P_{pa} stands for the purely atomic part

$$P_{\mathrm{pa}}(t) = \sum_{n=0}^{\infty} P_n \delta(t - r_n) \doteq \sum_{n=0}^{\infty} P_n \delta_{r_n}(t), \tag{2}$$

with elements $\{P_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$, ordered delays $0 = r_0 < r_1 < \cdots$, $\delta(t-r_n) = \delta_{r_n}(t)$ denoting the Dirac delta distribution centered in r_n, and $\sum_{n\geq 0}|P_n| < \infty$. The set $\mathscr{A}$ is a commutative convolution Banach algebra for the norm defined by

$$\|P\|_{\mathscr{A}} = \|P_{\mathrm{na}}\|_{L_1} + \sum_{n=0}^{\infty}|P_n|, \tag{3}$$

with unit element the Dirac delta distribution δ. Denoting $\hat{P}$ the Laplace transform of P, $\hat{\mathscr{A}}$ is the set of Laplace transforms of elements in $\mathscr{A}$. The set $\mathscr{A}^{p\times m}$ stands for the set of $p \times m$ matrices P with entries in $\mathscr{A}$, and the matrix norm $\|P\|_{\mathscr{A}} = \max_{i=1,\ldots,p}\sum_{j=1}^{m}\left\|P_{ij}\right\|_{\mathscr{A}}$, where P_{ij} denotes the (i,j) entry of P. Two matrices $N \in \mathscr{A}^{p\times m}$ and $D \in \mathscr{A}^{m\times m}$ are said to be right-coprime if there exist $X \in \mathscr{A}^{m\times p}$ and $Y \in \mathscr{A}^{m\times m}$ such that

$$XN + YD = I_m\delta, \tag{4}$$

where I_m stands for the $m \times m$ identity matrix. A pair (N, D) is a right-coprime factorization (r.c.f.) in $\mathscr{A}$ of a given plant P if D is non singular, $P = ND^{-1}$, and N, D are right-coprime over $\mathscr{A}$. Analogously, $(\widetilde{D}, \widetilde{N})$ is a left-coprime factorization (l.c.f.) in $\mathscr{A}$ of P if $\widetilde{D}$ is non singular, $P = \widetilde{D}^{-1}\widetilde{N}$, and $\widetilde{D} \in \mathscr{A}^{p\times p}$, $\widetilde{N} \in \mathscr{A}^{p\times m}$ are left coprime over $\mathscr{A}$, that is there exist $\widetilde{X} \in \mathscr{A}^{m\times p}$ and $\widetilde{Y} \in \mathscr{A}^{p\times p}$ such that $\widetilde{N}\widetilde{X} + \widetilde{D}\widetilde{Y} = I_p\delta$.

2.2. Properness, BIBO-stability and stabilization

Elements P in $\mathscr{A}$ are causal and proper, in the sense that $\sup_{\mathrm{Re}(s)\geq 0,\,|s|\geq r}|\hat{P}(s)|$ is bounded for some $r > 0$ (Curtain & Morris, 2009; Curtain & Zwart, 1995). Elements P_{na} in the ideal $L_1(0,\infty)$ of $\mathscr{A}$ are strictly proper, that is $\lim_{r\to\infty}\sup_{\mathrm{Re}(s)\geq 0,\,|s|\geq r}|\hat{P}_{\mathrm{na}}(s)|$ is zero (Curtain & Zwart, 1995, Property A.6.2 pp. 636). A matrix of transfer functions $\hat{P}$ is said to be proper (strictly proper) if its entries are proper (strictly proper). Let P be a $p \times m$ causal plant, and assume that P admits a r.c.f. (N, D), with $N \in \mathscr{A}^{p\times m}$ and $D \in \mathscr{A}^{m\times m}$. Using the Bézout identity (4), it is readily verified that P is proper if and only if D^{-1} is causal and proper. For $D = D_{\mathrm{pa}} + D_{\mathrm{na}}$, with $D_{\mathrm{pa}} = \sum_{i\geq 0} D_i\delta_{r_i}$, this in turn implies that D_0 is invertible in $\mathbb{R}^{m\times m}$.

A causal convolution system $y = Pu$ is said to be BIBO stable if $P \in \mathscr{A}^{p\times m}$ (Vidyasagar, 1985), in the sense that any bounded input yields a bounded output. In control perspective, let C be a controller with a l.c.f. $(\widetilde{D}_c, \widetilde{N}_c)$ in $\mathscr{A}^{m\times m} \times \mathscr{A}^{m\times p}$, interconnected with the plant P as represented in Fig. 1. The feedback system in Fig. 1 satisfies $\mathbf{y} = W_{P,C}\mathbf{u}$, where

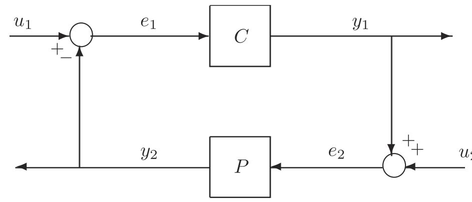

Fig. 1. Interconnection (P, C) by output feedback.

$$W_{P,C} = \begin{bmatrix} I_p\delta - N\Lambda^{-1}\widetilde{N}_c & -N\Lambda^{-1}\widetilde{D}_c \\ D\Lambda^{-1}\widetilde{N}_c & D\Lambda^{-1}\widetilde{D}_c \end{bmatrix}, \tag{5}$$

with $\mathbf{y} = \begin{bmatrix} y_1^T & y_2^T \end{bmatrix}^T$, $\mathbf{u} = \begin{bmatrix} u_1^T & u_2^T \end{bmatrix}^T$ and

$$\Lambda = \widetilde{N}_c N + \widetilde{D}_c D. \tag{6}$$

The pair (P, C) is said to be stable, or C stabilizes P, if $W_{P,C} \in \mathscr{A}^{(p+m)\times(p+m)}$. It is well known that (P, C) is stable if and only if Λ in (6) is a unit in $\mathscr{A}^{m\times m}$, that is Λ^{-1} exists and lies in $\mathscr{A}^{m\times m}$ (Quadrat, 2003; Vidyasagar, 1985).

As an illustration, an acoustic wave in a duct with Dirichlet boundary conditions and a boundary constant impedance writes (Curtain & Morris, 2009, Section 3.3 pp. 1106)

$$\hat{P}(s) = \frac{1 + \eta\,\mathrm{e}^{-rs}}{1 - \eta\,\mathrm{e}^{-rs}} \tag{7}$$

where η is the reflection coefficient such that $|\eta| < 1$. This proper and stable plant admits the coprime factorization in $\mathscr{A}$ described by

$$\hat{N}(s) = 1 + \eta\,\mathrm{e}^{-rs}, \quad \text{and} \quad \hat{D}(s) = 1 - \eta\,\mathrm{e}^{-rs}, \tag{8}$$

where $D = \delta - \eta\,\delta_r$ is a unit in $\mathscr{A}$. This coprime factorization satisfies (4) with

$$\hat{X}(s) = -\hat{Q}(s)\hat{D}(s), \qquad \hat{Y}(s) = \hat{D}(s)^{-1} + \hat{Q}(s)\hat{N}(s),$$

for any $Q \in \mathscr{A}$. Another example issued from Bonnet and Partington (2000) and the theory of transmission lines is

$$\hat{P}(s) = \frac{\mathrm{e}^{-\sqrt{s}}}{s-1} \tag{9}$$

with a simple pole at $s = 1$. A coprime factorization in $\mathscr{A}$ is $\hat{N}(s) = \frac{\mathrm{e}^{-\sqrt{s}}}{s+1}$ and $\hat{D}(s) = \frac{s-1}{s+1}$, which satisfies (4) for

$$\hat{X}(s) = -\frac{4\mathrm{e}}{s+1} \quad \text{and} \quad \hat{Y}(s) = \frac{(s+1)^2 - 4\mathrm{e}\mathrm{e}^{-\sqrt{s}}}{(s+1)(s-1)},$$

where it is noted that $\hat{Y}(s) \in \hat{\mathscr{A}}$ has a removable singularity at $s = 1$.

2.3. Graph topology and approximation

Graph topology gives a suitable framework for approximation in feedback perspective and robustness issues. We refer to Vidyasagar (1984, 1985), Zhu (1988) for details on this topology. For a causal plant P with a r.c.f. (N, D), $N \in \mathscr{A}^{p\times m}$, $D \in \mathscr{A}^{m\times m}$, a neighborhood of P in the graph topology can be characterized by convergence of coprime factorizations (Vidyasagar, 1985, ch. 7 p. 238). For this, we consider the following definition.