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High-precision frequency estimation of real sinusoids with reduced computational complexity using a model-based matched-spectrum approach

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ABSTRACT

Frequency-estimation algorithms devised for complex sinusoids, including the maximum-likelihood (ML) approach, when operating on real sinusoidal signals, suffer from spectral interference due to the superposition of the aliasing components at negative and positive frequencies. This paper introduces a frequency estimation ML-like algorithm, based on a spectral-matching approach, that avoids such superposition effect by incorporating it in the signal/spectrum model. As a result, the proposed method is able to generate a more precise frequency estimate in comparison to previous approaches at a comparable computational cost, as endorsed by provided computational analyses and simulation results.

1. Introduction

Frequency estimation is a standard problem in the signal processing field with a plethora of applications ranging from radar and satellite/mobile communications to general audio or speech processing and metrology [1–5]. The theoretical basis for the optimal frequency estimation, based on the maximum-likelihood (ML) criterion, of a discrete-time complex sinusoid embedded in noise was established in [6]. Later on, a series of algorithms based on the interpolation of the signal spectrum was devised to reduce the associated computational cost [7–15]. Generally speaking, all interpolation methods are computationally simple, requiring just a few operations in addition to the initial discrete Fourier transform (DFT) computation, and provide a very good ML approximation. They all, however, were initially devised for noisy complex-sinusoid signals, and, therefore, when dealing with real sinusoids, suffer from spectral superposition of the positive and negative frequency complex sinusoids, which introduces estimation bias and increases estimation variance.

To overcome such issues, a new frequency-estimation algorithm is considered based on a matched-spectrum concept, which correlates the measured DFT with the theoretical spectrum of a sampled sinusoid. The result is a new estimation method which yields very precise frequency estimates, particularly for high signal-to-noise ratios (SNR), at a reduced (comparable to the spectrum-interpolation algorithms) computational cost, as verified by computer simulations. In addition, both its estimation accuracy and robustness to noise can be scaled at the expense of increased computational complexity.

To introduce the proposed matched-spectrum (MS) method, this paper is organized as follows. Sections 2 and 3 revisit the ML and interpolation methods for frequency estimation, respectively. The proposed MS algorithm is introduced in Section 4, whereas Section 5 discusses some practical considerations on its implementation and performance. Section 6 includes some computational experiments illustrating the interesting results achieved by the proposed algorithm in comparison to previous schemes. Finally, Section 7 concludes the paper summarizing its technical contributions.

2. Maximum-likelihood estimation

Assume a complex sinusoid $s(n) = \tilde{a} \exp j(\tilde{\omega} n + \tilde{\theta})$, of amplitude $\tilde{a}$, frequency $\tilde{\omega}$, and phase $\tilde{\theta}$, is immersed in additive white Gaussian noise $v(n) = v_R(n) + j v_I(n)$, whose imaginary part $v_I(n)$ is the Hilbert transform of its real part $v_R(n)$. Assume also that

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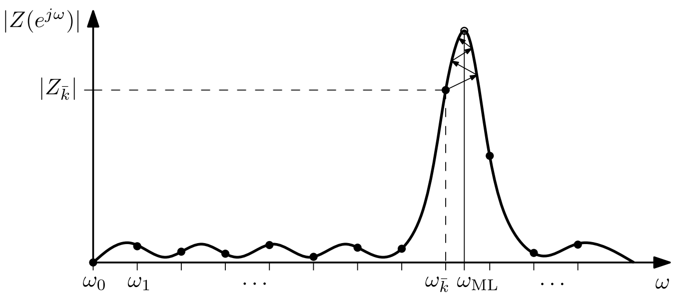

Fig. 1. ML frequency estimation ω_{ML} using the absolute value of the DFT: starting at the peak position $\omega_{\bar{k}}$ of $|Z(e^{j\omega_k})|$, estimate refinement is performed by numerical optimization of periodogram function, as given in Eq. (3).

$v(n)$ has zero mean and variance σ_v^2. The corresponding sample vector $\mathbf{z} = [z(0)z(1)\ldots z(N-1)]^{\mathrm{T}}$, where $z(n) = s(n) + v(n)$, has the joint distribution

$$f_{\mathbf{z}}(\mathbf{p}) = \left(\frac{1}{\sqrt{2\pi\sigma_v^2}}\right)^N e^{-\frac{1}{2\sigma_v^2}\sum_{n=0}^{N-1}[(x(n)-\mu_{\mathbf{p}}(n))^2+(y(n)-\nu_{\mathbf{p}}(n))^2]}, \qquad (1)$$

where $\mathbf{p} = [a, \omega, \theta]^{\mathrm{T}}$ is the parameter vector, $x(n)$ and $y(n)$ are the real and imaginary parts of $z(n)$, respectively, and

$$\mu_{\mathbf{p}}(n) = \tilde{a}\cos(\tilde{\omega}n + \tilde{\theta}), \qquad \nu_{\mathbf{p}}(n) = \tilde{a}\sin(\tilde{\omega}n + \tilde{\theta}). \qquad (2)$$

The maximum-likelihood (ML) parameter estimator $\mathbf{p}_{\mathrm{ML}} = [a_{\mathrm{ML}}, \omega_{\mathrm{ML}}, \theta_{\mathrm{ML}}]^{\mathrm{T}}$ of $\mathbf{p}$, given the observations $\mathbf{z}$, is the value of $\mathbf{p}$ that maximizes $f_{\mathbf{z}}(\mathbf{p})$ given in Eq. (1). After some algebraic development, one has that [6]:

- ω_{ML} is the value of ω that maximizes the periodogram

$$\left|A(\omega)\right| = \left|\frac{1}{N}\sum_{n=0}^{N-1} z(n)e^{-j\omega n}\right|. \qquad (3)$$

- θ_{ML} is the argument of $A(\omega_{\mathrm{ML}})$.
- $a_{\mathrm{ML}} = |A(\omega_{\mathrm{ML}})|$.

These results suggest a simple strategy for estimating the ML parameters:

1. Determine the discrete Fourier transform (DFT), $Z(e^{j\omega_k})$, of the sequence $\mathbf{z}$ and determine the (discrete) frequency value $\omega_{\bar{k}}$ associated with the maximum of its absolute value [16].
2. Starting at $\omega_{\bar{k}}$, use some numerical optimization algorithm to maximize $|A(\omega)|$ in Eq. (3) to determine ω_{ML} [6], as illustrated in Fig. 1.
3. Once ω_{ML} is estimated, compute $\theta_{\mathrm{ML}} = \arg\{A(\omega_{\mathrm{ML}})\}$ and $a_{\mathrm{ML}} = |A(\omega_{\mathrm{ML}})|$, as indicated above.

3. Fine adjustment by interpolation

Obtaining the ML estimate by optimizing the periodogram, besides being a cumbersome procedure due to the nature of the function evaluated at each iteration, may also not always converge to the desired solution, as analyzed in [17]. An alternative approach, which is quite simple and robust, is based on the interpolation of the DFT mainlobe points, which enables us to estimate the frequency deviation δ such that $\omega_{\mathrm{ML}} = \omega_{\bar{k}} + \delta$, as indicated in Fig. 2.

Among the several interpolation-based approaches found in the literature [7–15], one of the most successful employs [13]

$$\frac{\hat{\delta}}{\Delta} = \frac{\sqrt{1+8\gamma^2}-1}{4\gamma}, \qquad (4)$$

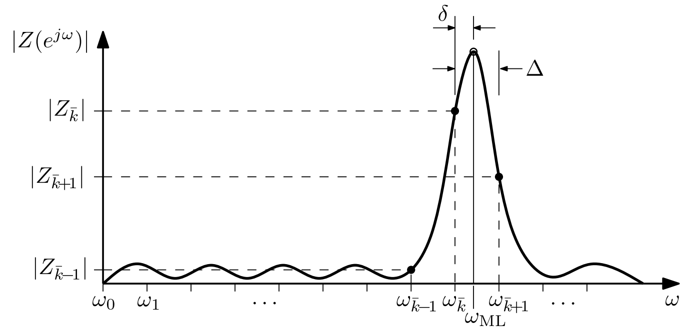

Fig. 2. ML frequency estimation ω_{ML} using the absolute value of the DFT: starting at the peak position $\omega_{\bar{k}}$ of $|Z(e^{j\omega_k})|$, a frequency deviation δ is estimated by interpolation of side points at $\omega_{\bar{k}-1}$ and $\omega_{\bar{k}+1}$.

where $\Delta = \frac{2\pi F_s}{N}$ is the DFT frequency resolution (with the sampling frequency F_s in samples per second), as represented in Fig. 2, and

$$\gamma = \frac{R_{-1} - R_1}{2R_0 + R_{-1} + R_1}, \qquad (5)$$

with R_k, for $k \in \mathbb{Z}$, being defined as

$$R_k = \mathrm{real}\left\{Z\left(e^{j\omega_{\bar{k}+k}}\right) \times \mathrm{conj}\left\{Z\left(e^{j\omega_{\bar{k}}}\right)\right\}\right\}, \qquad (6)$$

where $\mathrm{conj}\{\cdot\}$ denotes the complex-conjugate operation. An extension of this interpolator which employs four neighboring points and the periodogram peak at $\omega_{\bar{k}}$ is also presented in [13], with slightly superior computational complexity and estimation performance.

All these complex-sinusoid based algorithms, including the recent proposal given in [14], suffer from spectral leakage due to the spectral component centered at $\omega = -\tilde{\omega}$. This generates some bias on the final frequency estimate of real-sinusoid signals. The proposed method attempts to prevent this issue using a model-based approach, as described in the following section.

4. Proposed method: matched spectrum

The proposed matched-spectrum (MS) scheme attempts to match the DFT of the observation data, as determined in the first stage of all previous algorithms, to the theoretical spectrum $S_{\tilde{\omega},\tilde{\theta}}(e^{j\omega_k})$ of a frequency-$\tilde{\omega}$ and phase-$\tilde{\theta}$ sinusoid sampled at the same DFT frequency values ω_k. For that matter, one searches for the optimal values $\tilde{\omega}$ of ω and $\tilde{\theta}$ of θ that maximize the correlation

$$\tilde{R}_{k_0}(\omega, \theta) = \frac{\sum_{k=-k_0}^{k_0} Z(e^{j\omega_{\bar{k}+k}}) \times \mathrm{conj}\{S_{\omega,\theta}(e^{j\omega_{\bar{k}+k}})\}}{\sqrt{\sum_{k=-k_0}^{k_0} S_{\omega,\theta}(e^{j\omega_{\bar{k}+k}}) \times \mathrm{conj}\{S_{\omega,\theta}(e^{j\omega_{\bar{k}+k}})\}}}, \qquad (7)$$

where k_0 is the interval of interest around $\bar{k}$. In practice, this correlation function has the following interesting properties:

- In the noiseless case, it has a global maximum at $\omega = \tilde{\omega}$ and $\theta = \tilde{\theta}$.
- When $\omega \approx \tilde{\omega}$ and $\theta \approx \tilde{\theta}$, it does not present local minima, even in the presence of noise, as illustrated in Section 6, allowing a simple line-search procedure to determine $\tilde{\omega}$ and $\tilde{\theta}$ with a very high precision.
- Its value is readily approximated even for small values of k_0, including the trivial case $k_0 = 1$.
- For given values of ω and θ, its evaluation requires only $2(2k_0+1)$ complex multiplications and a single complex division (see detailed algorithm at the end of this section), in contrast to the periodogram function defined in Eq. (3), whose complexity is linear with the number of signal samples N.