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Sampling expansion for irregularly sampled signals in fractional Fourier transform domain

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ABSTRACT

Real-world signals are often not band-limited, and in many cases of practical interest sampling points are not always measured regularly. The purpose of this paper is to propose an irregular sampling theorem for the fractional Fourier transform (FRFT), which places no restrictions on the input signal. First, we construct frames for function spaces associated with the FRFT. Then, we introduce a unified framework for sampling and reconstruction in the function spaces. Based upon the proposed framework, an FRFT-based irregular sampling theorem without band-limiting constraints is established. The theoretical derivations are validated via numerical results.

1. Introduction

The fractional Fourier transform (FRFT) [1,2] is a generalization of the Fourier transform (FT) with additional free angle parameters. It can be interpreted as a rotation by an angle α in the time-frequency plane [3]. The FRFT can extend the utilities of the FT and has received much attention in recent years due to its numerous applications [1–9], including quantum physics, optics, radar, communications, signal processing, etc.

The FRFT of a function $f(t) \in L^2(\mathbb{R})$ is defined as [2]

$$F_\alpha(u) = \mathcal{F}^\alpha\{f(t)\}(u) \triangleq \int_{\mathbb{R}} f(t)\mathcal{K}_\alpha(u,t)dt \tag{1}$$

where $\mathcal{F}^\alpha$ denotes the FRFT operator, and kernel $\mathcal{K}_\alpha(u,t)$ is given by

$$\mathcal{K}_\alpha(u,t) = \begin{cases} A_\alpha e^{j\frac{u^2+t^2}{2}\cot\alpha} e^{-jtu\csc\alpha}, & \alpha \neq k\pi \\ \delta(t-u), & \alpha = 2k\pi \\ \delta(t+u), & \alpha = (2k-1)\pi \end{cases} \tag{2}$$

where $A_\alpha = \sqrt{(1-j\cot\alpha)/2\pi}$ and $k \in \mathbb{Z}$. Conversely, the inverse FRFT with respect to angle α is the FRFT with angle $-\alpha$. In general, we only consider the case of $0 < \alpha < \pi$, since (1) can easily be extended outside the interval $(0,\pi)$ by noting that $\mathcal{F}^{2\pi k}$ is the identity operator for any integer k and $\mathcal{F}^\alpha$ has the additivity property $\mathcal{F}^{\alpha+\beta}\{f(t)\} = \mathcal{F}^\alpha\{\mathcal{F}^\beta\{f(t)\}\}$. Note that when $\alpha = \pi/2$, (1) reduces to the FT.

Sampling theory plays a crucial role in signal processing and communications, which allows real-life signals in the continuous domain to be acquired, represented, and processed in the discrete domain. In the sense of the FRFT [10–21], the most classical sampling result is the theorem of Xia [10], which states that for a finite energy $\pi\sin\alpha$-fractional band-limited signal $f(t)$, i.e., signal $f(t) \in L^2(\mathbb{R})$ whose FRFT has support in $[-\pi\sin\alpha, \pi\sin\alpha]$,

$$f(t) = \sum_{n\in\mathbb{Z}} f[n]\,\mathrm{sinc}(t-n)e^{-j\frac{t^2-n^2}{2}\cot\alpha} \tag{3}$$

where $\mathrm{sinc}(\cdot) \triangleq \sin\pi(\cdot)/\pi(\cdot)$, and a normalized sampling step is used. Mathematically, Xia's sampling procedure is equivalent to computing the orthogonal projection of the input signal $f(t)$ on to the space $\mathcal{B}_\alpha = \overline{\mathrm{span}}\{\mathrm{sinc}(t-n)e^{-j\frac{t^2-n^2}{2}\cot\alpha}\}_{n\in\mathbb{Z}}$ of fractional band-limited signals. Unfortunately, the procedure is not appropriate for non-bandlimited signals. However, if we substitute $\phi(t)$ for sinc(t) in the space $\mathcal{B}_\alpha$, then $\mathcal{B}_\alpha$ is exactly the function space associated with the FRFT [22], i.e.,

$$\mathcal{V}_\alpha(\phi) = \overline{\mathrm{span}}\left\{\phi_{n,\alpha}(t) \triangleq \phi(t-n)e^{-j\frac{t^2-n^2}{2}\cot\alpha}\right\}_{n\in\mathbb{Z}} \subset L^2(\mathbb{R}). \tag{4}$$

Realizing this property, Shi et al. [22] extended (3) to space $\mathcal{V}_\alpha(\phi)$. Specifically, let $\phi(t)$ be a continuous function in $L^2(\mathbb{R})$ such that $\{\phi_{n,\alpha}(t)\}_{n\in\mathbb{Z}}$ is a Riesz basis for space $\mathcal{V}_\alpha(\phi)$ and $\{\phi[n]\}_{n\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$. There exists a function $s(t) \in L^2(\mathbb{R})$ with $s(t)e^{-j\frac{t^2}{2}\cot\alpha} \in \mathcal{V}_\alpha(\phi)$ such that [22]

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$$f(t)=\sum_{n\in\mathbb{Z}} f[n]s(t-n)e^{-j\frac{t^2-n^2}{2}\cot\alpha} \quad (5)$$

holds in $L^2(\mathbb{R})$ for any $f(t)\in\mathcal{V}_\alpha(\phi)$ if and only if $\frac{1}{\sqrt{2\pi}\widetilde{\Phi}(u\csc\alpha)}\in L^2[0,2\pi\sin\alpha]$ holds, where $\widetilde{\Phi}(u\csc\alpha)$ denotes the discrete-time FT (with its argument scaled by $\csc\alpha$) of $\phi[n]$. It is clear that (5) applies only to the case of regular sampling. In many real applications, sampling points are not always measured regularly. Sometimes sampling steps need to be fluctuated according to input signals so as to reduce the number of samples as well as computational complexity. There are also many cases where undesirable jitter exists in sampling instants. Some communication systems may suffer from random delay due to channel traffic congestion and encoding delay. In such cases, the sampling system will become more efficient when a perturbation factor is considered. Towards this end, Zhao et al. [23] introduced an extension of Paley–Wiener's $\frac{1}{4}$-Theorem based upon the linear canonical transform, which is a generalization of the FRFT. However, the extension is appropriate to the case of band-limited signals only. Therefore, it is desirable to derive an irregular sampling theorem for the FRFT without band-limiting constraints. The purpose of the present paper is to fill this gap by exploiting the theory of frames. We first construct frames for function spaces associated with the FRFT, and then propose a unified framework for sampling and reconstruction in the function spaces. Further, without band-limited assumption, we establish an irregular sampling theorem for the FRFT. Numerical results are also presented.

The remainder of this paper is organized as follows. In Section 2, notation is introduced, and some facts of the frame theory and the discrete-time FRFT are briefly reviewed. In Section 3, a unified framework for FRFT-based sampling and reconstruction in function spaces is proposed. Then, an irregular sampling theorem of the FRFT without band-limiting constraints is established in Section 4. In Section 5, numerical results are given. Finally, concluding remarks are drawn in Section 6.

2. Preliminaries

Continuous signals are denoted with parentheses, e.g., $f(t)$, $t\in\mathbb{R}$, and discrete signals with brackets, e.g., $q[n]$, $n\in\mathbb{Z}$. The scalar product of two functions $f(t)$ and $g(t)$ in $L^2(\mathbb{R})$ is defined as $\langle f,g\rangle=\int_{\mathbb{R}} f(t)g^*(t)dt$, where $*$ in the superscript denotes the complex conjugate. The norm of a function $f(t)\in L^2(\mathbb{R})$ is defined as $\|f\|=\langle f,f\rangle^{1/2}$. For a measurable function $g(t)$ on $\mathbb{R}$, let $\|g\|_\infty=\operatorname{ess\,sup}|g(t)|$ and $\|g\|_0=\operatorname{ess\,inf}|f(t)|$ be the essential supremum and infimum of $|g(t)|$, respectively. The characteristic function of a measurable subset $E\subset\mathbb{R}$ is denoted with $\chi_E(t)$, where $\chi_E(t)=1$, $t\in E$, and 0 otherwise.

A function sequence $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ is said to be a frame if there exists a constant $C\geq 1$ such that

$$C^{-1}\|f\|^2\leq\sum_{n\in\mathbb{Z}}\left|\left\langle f(t),\varphi_n(t)\right\rangle\right|^2\leq C\|f\|^2 \quad (6)$$

holds for any $f(t)\in\mathcal{H}$. If the removal of one element $\varphi_m(t)$ renders the function sequence $\{\varphi_n(t)\}_{n\neq m}$ no longer a flame, then it is called an exact frame. An exact frame is a Riesz basis. Obviously, a Riesz basis is also a frame [24]. For any frame $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ of $\mathcal{H}$, there exists a so-called dual frame $\{\tilde{\varphi}_n(t)\}_{n\in\mathbb{Z}}\subset\mathcal{H}$ such that

$$f(t)=\sum_{n\in\mathbb{Z}}\left\langle f(t),\tilde{\varphi}_n(t)\right\rangle\varphi_n(t)=\sum_{n\in\mathbb{Z}}\left\langle f(t),\varphi_n(t)\right\rangle\tilde{\varphi}_n(t) \quad (7)$$

holds in $L^2(\mathbb{R})$ for any $f(t)\in\mathcal{H}$. Take a linear operator T on $\mathcal{H}$ defined as

$$T\left\{f(t)\right\}=\sum_{n\in\mathbb{Z}}\left\langle f(t),\varphi_n(t)\right\rangle\varphi_n(t). \quad (8)$$

Then, $\langle T\{f(t)\},f(t)\rangle=\sum_{n\in\mathbb{Z}}|\langle f(t),\varphi_n(t)\rangle|^2$. Eq. (6) implies that the operator T is bounded, self-conjugate, and invertible. It is easy to see that the function sequence $T^{-1}\{\varphi_n(t)\}_{n\in\mathbb{Z}}$ is a dual frame of frame $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$, and T is called a frame transform of $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$. The scalar sequence $\{\langle f(t),\varphi_n(t)\rangle\}_{n\in\mathbb{Z}}$ is called a moment sequence of $f(t)$ to frame $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$. Let $f(t)=\sum_{n\in\mathbb{Z}}c_n\varphi_n(t)$. If $\{c_n\}_{n\in\mathbb{Z}}$ is a moment sequence of a function to $\{\varphi_n(t)\}_{n\in\mathbb{Z}}$, then it must be

$$c_n=\left\langle T^{-1}\left\{f(t)\right\},\varphi_n(t)\right\rangle,\quad\forall n\in\mathbb{Z}. \quad (9)$$

This follows from the fact that $c_n=\langle h,\varphi_n(t)\rangle$ for some function $h(t)\in\mathcal{H}$, and $T^{-1}\{f(t)\}=\sum_{n\in\mathbb{Z}}\langle h(t),\varphi_n(t)\rangle T^{-1}\{\varphi_n(t)\}=h(t)$ in $L^2(\mathbb{R})$.

There are two different definitions [12,25] for the discrete-time FRFT (DTFRFT) in the literature. We adopt the one introduced in [12], which has a simple structure. The DTFRFT of a sequence $\{q[n]\}_{n\in\mathbb{Z}}$ is defined as [12]

$$\widetilde{Q}_\alpha(u)=\sum_{n\in\mathbb{Z}}q[n]\mathcal{K}_\alpha(u,n). \quad (10)$$

Note that if $\{q[n]\}_{n\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$, $\widetilde{Q}_\alpha(u)\in L^2[0,2\pi\sin\alpha]$. Conversely, the inverse DTFRFT is given by $q[n]=\int_I\widetilde{Q}_\alpha(u)\mathcal{K}_\alpha^*(u,n)du$, where $I\triangleq[0,2\pi\sin\alpha]$. The DTFRFT has the following chirp-periodicity [22]:

$$\widetilde{Q}_\alpha(u+2\pi\sin\alpha)e^{-j\frac{(u+2\pi\sin\alpha)^2}{2}\cot\alpha}=\widetilde{Q}_\alpha(u)e^{-j\frac{u^2}{2}\cot\alpha}. \quad (11)$$

3. A unified framework for sampling and reconstruction in function spaces associated with the FRFT

For a continuous-time function $\phi(t)\in L^2(\mathbb{R})$, define

$$G_{\phi,\alpha}(u)\triangleq\sum_{k\in\mathbb{Z}}\left|\Phi(u\csc\alpha+2k\pi)\right|^2 \quad (12)$$

where $\Phi(u\csc\alpha)$ denotes the FT (with its argument scaled by $\csc\alpha$) of $\phi(t)$. Generally speaking, the function sequence $\{\phi_{n,\alpha}(t)\}_{n\in\mathbb{Z}}$ with the form defined in (4) is not a Riesz basis for $\mathcal{V}_\alpha(\phi)$. In fact, it is a Riesz basis for $\mathcal{V}_\alpha(\phi)$ if and only if [21,22]

$$0\leq\left\|G_{\phi,\alpha}(u)\right\|_0\leq\left\|G_{\phi,\alpha}(u)\right\|_\infty<\infty. \quad (13)$$

In this case, $\phi(t)$ is said to be a stable generator for $\mathcal{V}_\alpha(\phi)$. Moreover, the function sequence $\{\phi_{n,\alpha}(t)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathcal{V}_\alpha(\phi)$ if and only if $G_{\phi,\alpha}(u)=1$ holds for almost everywhere $u\in\mathbb{R}$. In particular, if we choose $\phi(t)=\mathrm{sinc}(t)$, the stable space $\mathcal{V}_\alpha(\mathrm{sinc})$ is exactly the space of all $\pi\sin\alpha$-fractional band-limited signals with finite energy. Sampling in $\mathcal{V}_\alpha(\mathrm{sinc})$ leads to Xia's sampling theorem of the FRFT for regularly sampled signals [10]. In this paper, we will present an irregular sampling theorem of the FRFT in $\mathcal{V}_\alpha(\phi)$ for a general stable generator $\phi(t)$. We also need $\phi(t)=O((1+|t|)^{-\epsilon})$ for some $\epsilon>1/2$. For any $f(t)\in\mathcal{V}_\alpha(\phi)$, it follows from (4) that

$$f(t)=\sum_{m\in\mathbb{Z}}c[m]\phi(t-m)e^{-j\frac{t^2-m^2}{2}\cot\alpha} \quad (14)$$

where $\{c[m]\}_{m\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$. It is easy to see that

$$\left|\sum_{m\in\mathbb{Z}}c[m]\phi(t-m)e^{-j\frac{t^2-m^2}{2}\cot\alpha}\right|^2\leq\sum_{m\in\mathbb{Z}}\left|c[m]\right|^2\sum_{m\in\mathbb{Z}}\left|\phi(t-m)\right|^2 \quad (15)$$

which implies that the series defined in (14) point-wise converges to a continuous function in $\mathcal{V}_\alpha(\phi)$. Without loss of generality, we can take any $f(t)\in\mathcal{V}_\alpha(\phi)$ as a continuous function.