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Brief paper

Distributed constrained optimal consensus of multi-agent systems[☆]

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ABSTRACT

We study a distributed optimal consensus problem of continuous-time multi-agent systems with a common state set constraint. Each agent is assigned with an individual cost function which is coercive and convex. A distributed control protocol is to be designed to guarantee a consensus, and in the meanwhile reach the minimizer of the aggregate cost functions within the constraint set. Three terms are included in the protocol: local averaging, local projection, and local subgradient with a diminishing but persistent gain. It is shown that the constrained optimal consensus can be achieved under a uniformly jointly connected communication network with bounded time-varying edge weights.

1. Introduction

As a basic problem in distributed cooperative control for multi-agent systems, consensus has gained much attention from researchers in recent years. The objective of distributed consensus is to reach an agreement of a certain variable of common interest by local information exchange. Many algorithms have been proposed and found widespread applications in coordination, rendezvous, flocking, source localization, etc. (Chai, Lin, Lin, & Zhang, 2014; Jadbabaie, Lin, & Morse, 2003; Lin, Morse, & Anderson, 2003; Olfati-Saber, 2006).

In many practical problems, consensus with constraints are to be considered. The constraints include state constraint (Sun, Jin Ong, & White, 2013), motion constraint of maximum speed and acceleration (Lin et al., 2003), and network constraint to stay connected (Zavlanos & Pappas, 2008). By decomposition and incremental subgradient methods, Johansson, Speranzon, Johansson, and Johansson (2008) dealt with the consensus problem subject to convex input constraints and linear state constraints. With model predictive control of one-step horizon, Franceschelli, Egerstedt, Giua, and Mahulea (2009) were able to drive a network of agents to their centroid while staying connected and satisfying the motion constraint. Another focus is on constraint of the final consensus value. Using projection, Shi, Johansson, and Hong (2013) computed the consensus point within the non-empty intersection of different convex sets. A similar problem was addressed in Lee and Mesbahi (2011) via logarithmic barrier functions.

On the other hand, the consensus problem becomes a distributed optimization problem when the consensus value is required to minimize the sum of individual convex functions. Many works are based on subgradient method. Nedić and Ozdaglar (2009) obtained an approximate optimal solution with a common constant step size about the local subgradient. A projected subgradient method was proposed in Nedic, Ozdaglar, and Parrilo (2010) to deal with a set constraint, where the subgradient step is firstly taken on the local average for an intermediate estimate, and then the estimate is projected onto the common constraint for an update of the state. It was later extended to the dual problem with inequality and equality constraints in Yuan, Xu, and Zhao (2011) and Zhu and Martinez (2012). Note that all the above works are in discrete-time.

As for continuous-time case, fewer results have been obtained. Conditions to guarantee the convergence of distributed unconstrained convex optimization for strongly connected graphs were examined in Shi, Proutiere, and Johansson (2012). By including a quadratic penalty in the Lagrangian problem, Wang and Elia (2010, 2011) applied saddle-point dynamics to reach the unconstrained minimizer of the sum of differential cost functions under a fixed undirected topology, which was extended later to the case of non-smooth convex functions (Gharesifard & Cortes, 2014). Compared

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with the discrete-time case, a continuous-time formulation enables to employ more techniques and arguably provide more insight due to its independence of particular realization. Still, there is much to be explored in applying distributed optimizing algorithms to continuous-time systems.

In this paper, we study a consensus problem considering both constraint and optimization: the distributed constrained optimal consensus for continuous-time multi-agent systems with a common convex set constraint. A motivational example comes from the constrained rendezvous of unmanned aerial vehicles (UAVs) within a prescribed safety area, when the choice of the rendezvous location depends on minimizing the aggregate distance from the starting points to the final location. Each node is assumed to be a continuous integrator system, and assigned with a local coercive convex cost function. A distributed control input for each agent is to be designed to attain a consensus value minimizing the sum of local functions over the constraint set. To this end, three terms are incorporated into the control input: a local averaging term to guarantee the asymptotic consensus, a projection term to make the state values approach the constraint set, and a subgradient term with a diminishing but persistent gain to drive the consensus state to the constrained minimum. Due to the non-uniqueness of subgradient, the algorithm is modeled as a differential inclusion, and the solving of the constrained optimal consensus problem relies on analyzing the limit set of its solution with non-smooth techniques.

The contributions of this paper are as follows. Compared with the discrete-time work (Nedic et al., 2010), we modify the projection term such that the state is to approach the constraint set rather than to be directly projected onto it. We also relax the condition about the diminishing scaling on subgradient without the extra requirement of being square-integrable. Besides, the bounded subgradient assumption can be completely removed under the mild assumption of coercive cost functions. And when compared with other continuous-time works (Shi et al., 2012), we are able to deal with more general non-smooth cost functions.

The rest of the paper is organized as follows. Some preliminaries are recalled in Section 2. The problem formulation is stated in Section 3, with the main result presented. The complete convergence analysis is conducted in Section 4. Numerical examples and concluding remarks are respectively provided in Sections 5 and 6.

Some notations and abbreviations are used throughout this paper. $\mathcal{R}$ and $\mathcal{R}_{\geq 0}$ respectively denote the real and nonnegative real numbers. $\mathbf{1}_n$ is an n dimensional vector of ones and $J = \frac{1}{n}\mathbf{1}_n\mathbf{1}_n'$. $\langle x, y\rangle$ and $|x-y|$ are standard inner product and the corresponding distance respectively for $x, y \in \mathcal{R}^n$. $[M]_{i,j}$ denotes the (i, j)-th entry of $M \in \mathcal{R}^{m\times n}$ and M' its transpose, while $\|M\| = \sup_{|x|\neq 0} |Mx|/|x|$ for a square matrix M. $X \otimes Y$ is the Kronecker product of X and Y. $\lceil a\rceil$ is the smallest integer not less than a. $\sum_i$ denotes a summation for all possible index i, which similarly applies to $\max_i$. The terms "upper semi-continuous", "absolutely continuous" and "almost everywhere" are abbreviated as u.s.c., a.c. and a.e., respectively.

2. Preliminaries

Some preliminaries, as well as main lemmas used in the analysis, are reviewed in this section.

2.1. Preliminaries of graph theory

A multi-agent system can be modeled as an undirected graph $\mathcal{G}$, which consists of a node set $\mathcal{V} = \{1, \dots, n\}$ and an edge set of unordered pairs $\mathcal{E} = \{(i, j) : i, j \in \mathcal{V}\}$ excluding self-loop (i, i). $(i, j) \in \mathcal{E}$ indicates a mutual communication between node i and node j. $\mathcal{N}_i = \{j \mid j \in \mathcal{V},\ (j, i) \in \mathcal{E}\}$ is the neighbor set of node i. A path from node l_0 to node l_d is defined by $(l_0, l_1), \dots, (l_{d-1}, l_d) \in \mathcal{E}(\mathcal{G})$, where $l_0, \dots, l_d$ are distinct nodes. $\mathcal{G}$ is connected if a path exists for any pair of different nodes. Moreover, a symmetric matrix $A \in \mathcal{R}^{n\times n}_{\geq 0}$ is used to represent the weights on the edges, and $[A]_{i,j} > 0$ iff $(j, i) \in \mathcal{E}$. The triplet $\{\mathcal{V}, \mathcal{E}, A\}$ completely describes the weighted graph $\mathcal{G}$. Conversely, given a symmetric matrix $P \in \mathcal{R}^{n\times n}_{\geq 0}$, an undirected graph $\mathcal{G}(P)$ can be associated by letting $\mathcal{V} = \{1, \dots, n\}$, $\mathcal{E} = \{(i, j) : [P]_{i,j} > 0, i \neq j\}$ and weight matrix P_d with $[P_d]_{i,j} = \begin{cases}[P]_{i,j}, & i \neq j;\\ 0, & i = j.\end{cases}$ For a graph $\mathcal{G}$, the Laplacian matrix $L = D_{\mathcal{G}} - A$ is useful to algebraically examine the connectivity, where $D_{\mathcal{G}}$ is a diagonal matrix with the ith diagonal entry as $D_i \stackrel{\Delta}{=} \sum_{j\in\mathcal{N}_i} a_{ij}$. If $\mathcal{G}$ is undirected, L has the Jordan decomposition $L = T_L \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}T_L'$ with the unitary matrix $T_L = [\frac{\mathbf{1}_n}{\sqrt{n}}\ \phi_2 \dots \phi_n]$, where $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ and $\mathcal{G}$ is connected iff $\lambda_2 > 0$ (Godsil & Royle, 2001).

We use the switching signal $\sigma : [t_0, +\infty) \to \mathcal{Q}$ to define the time-varying communication, with the set $\mathcal{Q}$ comprising of all the possible undirected weighted graphs assuming an identical node set $\mathcal{V}$. The union graph $\mathcal{G}_\sigma[t_1, t_2)$ over the time interval $[t_1, t_2)$ is jointly connected, if $\{\mathcal{V}, \bigcup_{t\in[t_1,t_2)} \mathcal{E}(\sigma(t))\}$ is connected. The same notation also applies to the discrete-time case, where $[t_1, t_2)$ stands for a sequence of consecutive time instants $\{t_1, t_1 + 1, \dots, t_2 - 1\}$. If there exists a constant $T > 0$ such that $\mathcal{G}_\sigma[t, t + T)$ is jointly connected for any t, $\mathcal{G}_\sigma$ is called uniformly jointly connected, where the subscript σ may be not specified.

2.2. Convex sets and convex functions

Given $x \in \mathcal{R}^m$ and $C \subseteq \mathcal{R}^m$, $|x|_C = \inf_{c\in C} |x - c|$ is the distance from x to C. For a closed convex set $\mathcal{C} \in \mathcal{R}^m$, $P_{\mathcal{C}}(x) \in \mathcal{C}$ is the projection of x onto $\mathcal{C}$, uniquely satisfying $|x - P_{\mathcal{C}}(x)| = |x|_{\mathcal{C}}$, and we have

$$\langle P_{\mathcal{C}}(x) - x, P_{\mathcal{C}}(x) - y\rangle \leq 0, \quad \forall y \in \mathcal{C}. \tag{1}$$

The following lemma estimates the inner product involving the projection vector, which may be seen as the counterpart of Cauchy–Schwarz inequality in the convex context. It can also be found in Shi and Hong (2009, Lemma 13), and here we provide a more concise and intuitive proof.

Lemma 1. *Given $\mathcal{C} \subset \mathcal{R}^m$ closed and convex, we have*

$$\langle x - P_{\mathcal{C}}(x), y - x\rangle \leq |x|_{\mathcal{C}}(|y|_{\mathcal{C}} - |x|_{\mathcal{C}}), \quad \forall x, y. \tag{2}$$

Proof. We only need to discuss the case of $|x|_{\mathcal{C}} \neq 0$. Define $r = \frac{1}{|x|_{\mathcal{C}}}(x - P_{\mathcal{C}}(x))$ and $H = \{z : \langle r, z - P_{\mathcal{C}}(x)\rangle = 0\}$ as the hyperplane supporting $\mathcal{C}$ at $P_{\mathcal{C}}(x)$. Clearly $\mathcal{C} \subset H^- = \{z : \langle r, z - P_{\mathcal{C}}(x)\rangle \leq 0\}$. Now let $y_H = \langle r, y - P_{\mathcal{C}}(x)\rangle$. Noticing that $\langle r, y - x\rangle = \langle r, y - P_{\mathcal{C}}(x) + P_{\mathcal{C}}(x) - x\rangle = y_H - x_H$ and $x_H = |x|_{\mathcal{C}}$, the proof is completed by observing that $y_H = d(y, H) \leq |y|_{\mathcal{C}}$ when $y_H \geq 0$. □

The subdifferential of a convex function f at x is the set

$$\partial f(x) = \{s : f(y) \geq f(x) + \langle s, y - x\rangle,\ \forall y\},$$

with the element $s \in \partial f(x)$ called a subgradient of f at x. $\partial f(x)$ is a nonempty compact convex set for each x and the set-valued map ∂f is u.s.c. everywhere. The subdifferential of $\varphi(t) = f(x+t(y-x))$ can be found in the lemma below, reminiscent of the directional derivative for a differentiable f:

Lemma 2 (*Hiriart-Urruty & Lemaréchal, 2001*)**.** *$\partial\varphi(t) = \{\langle s, y - x\rangle : s \in \partial f(x_t)\}$; or symbolically $\partial\varphi(t) = \langle\partial f(x_t), y - x\rangle$.*