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Brief paper

On potential equations of finite games☆

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ABSTRACT

In this paper, some new criteria for detecting whether a finite game is potential are proposed by solving potential equations. The verification equations with the minimal number for checking a potential game are obtained for the first time. Compared with the existing results, a reduced-complexity testing condition is derived. Some connections between the potential equations and the existing characterizations of potential games are established. It is revealed that a finite game is potential if and only if its every bi-matrix sub-game is potential.

1. Introduction

Game theory, the science of strategic decision making pioneered by John von Neumann (see von Neumann & Morgenstern, 1953), has wide real-world applications in many fields, including economics, biology, computer science and engineering. The Nash equilibrium, named after John Forbes Nash, Jr., is a fundamental concept in game theory, which represents stable states of complex systems such as economic systems, transportation networks and wireless communication networks. The existence and computing of Nash equilibria are two central issues in the theory of games. For two-player zero-sum games, von Neumann proved the existence of mixed-strategy equilibria using Brouwer Fixed Point Theorem. Nash proved that if mixed strategies are allowed, then every game with a finite number of players and strategies has at least one Nash equilibrium (Nash, 1951). Although pure strategies are conceptually simpler than mixed strategies, it is usually difficult to guarantee the existence of a pure-strategy Nash equilibrium (PNE). However, it is shown that every finite potential game possesses a PNE (Monderer & Shapley, 1996). The idea of potential functions was firstly proposed by Rosenthal (1973). A game is said to be a potential game if it admits a potential function. The incentive of all players to change their strategy can be expressed by the difference in values of the potential function. For a potential game, a PNE can be found by searching the maximal values of the potential function. PNEs are very important for many practical complex control systems such as multi-agent systems (Marden, Arslan, & Shamma, 2009) and wireless networks (Candogan, Menache, Ozdaglar, & Parrilo, 2010; Moragrega, Closas, & Ibars, 2015) since in practice the desirable operating point is just a PNE. Moreover, it is very efficient to design a complex control system as a potential game to guarantee the existence of PNEs and the corresponding potential function is usually utilized to design a distributed control algorithm to ensure the convergence to the desirable operating point.

An important problem is how to check whether a game is a potential game. Monderer and Shapley (1996) first proposed necessary and sufficient conditions for potential games. But it is required to verify all the simple closed paths with length 4 for any pair of players. Then Hino (2011) gave an improved condition for detecting potential games, which has a lower complexity than that of Monderer and Shapley (1996) due to that only the adjacent pairs of strategies of two players are needed to check. In Ui (2000), it is proved that a game is potential if and only if the payoff functions coincide with the Shapley value of a particular class of cooperative games indexed by the set of strategy profiles. Game decomposition is an important method for potential games (Candogan, Menache, Ozdaglar, & Parrilo, 2011; Hwang & Rey-Bellet, 2011; Sandholm, 2010) and some new necessary and sufficient conditions for detecting potential games are obtained. Sandholm (2010) established connections between his results and that in Ui (2000). But the number of the obtained verification

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equations is not the minimum, which is just an important problem to solve in this paper. In Sandholm (2010), it is proved that a finite game is a potential game if and only if, in each of the component games, all active players have identical payoff functions, and that in this case, the potential function can be constructed.

Recently, Cheng (2014) developed a novel method, based on the left semi-tensor product of matrices, to deal with games including potential games, networked games and evolutionary games (Cheng, 2014; Cheng, He, Qi, & Xu, 2015; Cheng, Xu, He, & Qi, 2014; Cheng, Xu, & Qi, 2014; Guo, Wang, & Li, 2013). In Cheng (2014), a linear system, called potential equation, is proposed, and then it is proved that a finite game is potential if and only if its potential equation is solvable. With a solution of the potential equation, the potential function can be directly calculated.

A natural question is how to establish the connection between the potential equation and some other existing criteria of potential games. Moreover, two more interesting problems are how to get the verification condition with minimum number of equations and how to reduce the computation complexity. In this paper, we further investigate the solvability of the potential equation. An equivalence transformation is constructed to convert the augmented matrix of the potential equation into the reduced row echelon form. Based on this technique, some new necessary and sufficient conditions for potential games are obtained. For potential games, a new formula to calculate the potential functions is proposed. Based on the obtained results, it is revealed the connection between the potential equation and the results in Hino (2011) and Sandholm (2010). Compared with existing references, this paper is in a more general frame without assuming that all the players have the same number of strategies. A testing condition with minimum number of equations for potential games is achieved and the computation complexity is essentially reduced.

Throughout the paper, we denote the $k \times k$ identity matrix by I_k, the ith column of I_k by δ_k^i, the n-dimensional column vector whose entries are all equal to 1 by $\mathbf{1}_k$, Kronecker product by $\otimes$ and the real number field by $\mathbb{R}$. For statement ease, we need some other notations as follows:

$$B_k := [I_{k-1}, \ -\mathbf{1}_{k-1}], \qquad D_k := [I_{k-1}, \ 0] \in \mathbb{R}^{(k-1)\times k}, \tag{1}$$

$$H_k = I_k - \frac{1}{k}\mathbf{1}_k\mathbf{1}_k^{\mathrm{T}}, \qquad k^{[p,q]} := \begin{cases} \prod_{j=p}^{q} k_j, & q \ge p, \\ 1, & q < p. \end{cases} \tag{2}$$

2. Preliminaries

Definition 1 (*Monderer & Shapley, 1996*). A *finite game* is a triple $\mathcal{G} = (\mathcal{N}, \ \mathcal{S}, \ \mathcal{C})$, where

(i) $\mathcal{N} = \{1, 2, \ldots, n\}$ is the set of players;
(ii) $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_n$ is the strategy set, where each $\mathcal{S}_i = \{s_1^i, s_2^i, \ldots, s_{k_i}^i\}$ is the strategy set of player i;
(iii) $\mathcal{C} = \{c_1, c_2, \ldots, c_n\}$ is the set of payoff functions, where every $c_i : \mathcal{S} \to \mathbb{R}$ is the payoff function of player i.

Let $c_{i_1 i_2 \cdots i_n}^{\mu} = c_\mu(s_{i_1}^1, s_{i_2}^2, \ldots, s_{i_n}^n)$ where $1 \le i_s \le k_s$ and $s = 1, 2, \ldots, n$. Then the finite game can be described by the arrays

$$C_\mu = \{c_{i_1 i_2 \ldots i_n}^{\mu} | 1 \le i_s \le k_s, \ s = 1, 2, \ldots, n\} \tag{3}$$

with $\mu = 1, 2, \ldots, n$. Particularly, for a 2-player game, the $k_1 \times k_2$ matrices $C_1 = (c_{ij}^1)$ and $C_2 = (c_{ij}^2)$ are the payoff matrices of players 1 and 2 respectively. Therefore, a 2-player finite game is also called a *bi-matrix game*, which is usually denoted by $\mathcal{G} = (C_1, C_2)$.

Definition 2 (*Monderer & Shapley, 1996*). A finite game $\mathcal{G} = (\mathcal{N}, \ \mathcal{S}, \ \mathcal{C})$ is said to be *potential* if there exists a function $p : \mathcal{S} \to \mathbb{R}$, called the *potential function*, such that $c_i(x, s^{-i}) - c_i(y, s^{-i}) = p(x, s^{-i}) - p(y, s^{-i})$ for all $x, y \in \mathcal{S}_i$, $s^{-i} \in \mathcal{S}^{-i}$ and $i = 1, 2, \ldots, n$, where $\mathcal{S}^{-i} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_{i-1} \times \mathcal{S}_{i+1} \times \cdots \times \mathcal{S}_n$.

Definition 3 (*Cheng, Qi, & Li, 2011*). Assume $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$. Let $\alpha = \mathrm{lcm}(n, p)$ be the least common multiple of n and p. The left semi-tensor product of A and B is defined as $A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{p}})$.

Since the left semi-tensor product is a generalization of the traditional matrix product, the left semi-tensor product $A \ltimes B$ can be directly written as AB. If the number of A's columns is equal to that of B's rows, then the left semi-tensor product AB is just the traditional product. Identifying each strategy s_j^i with the logical vector $\delta_{k_i}^j$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k_i$, Cheng (2014) gave a new expression of the payoff functions using the left semi-tensor product.

Lemma 4 (*Cheng, 2014*). *Let $x_i \in \mathcal{S}_i$ be any strategy expressed in the form of logical vectors. Then, for any payoff function c_i of a finite game $\mathcal{G}$ shown in Definition 1, there exists a unique row vector $V_i^c \in \mathbb{R}^n$ such that*

$$c_i(x_1, x_2, \ldots, x_n) = V_i^c x_1 x_2 \ldots x_n, \tag{4}$$

where V_i^c is called the structure vector of c_i.

Remark 5. It is easy to see that V_i^c is just the row vector composed of the elements of C_i in the lexicographic order (see (3)). Let $C = [(V_1^c)^{\mathrm{T}}, (V_2^c)^{\mathrm{T}}, \ldots, (V_n^c)^{\mathrm{T}}]^{\mathrm{T}}$. Then C is just the *payoff matrix* of $\mathcal{G}$ given by Cheng (2014).

In Cheng (2014), the potential equation is proposed in the case that $k_i = k$ for all $i = 1, 2, \ldots, n$. In Cheng, Liu, Zhang, and Qi (2015), the general potential equation is stated as follows:

$$\Psi \xi = b, \tag{5}$$

where

$$\Psi = \begin{bmatrix} -\Psi_1 & \Psi_2 & & & \\ -\Psi_1 & & \Psi_3 & & \\ \vdots & & & \ddots & \\ -\Psi_1 & & & & \Psi_n \end{bmatrix}, \qquad \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix},$$

$$b = \begin{bmatrix} (V_2^c - V_1^c)^{\mathrm{T}} \\ (V_3^c - V_1^c)^{\mathrm{T}} \\ \vdots \\ (V_n^c - V_1^c)^{\mathrm{T}} \end{bmatrix}$$

and $\Psi_i = I_{k^{[1,i-1]}} \otimes \mathbf{1}_{k_i} \otimes I_{k^{[i+1,n]}}$ for every $i = 1, 2, \ldots, n$.

Lemma 6 (*Cheng, 2014*; *Cheng, Liu et al., 2015*). *A finite game $\mathcal{G}$ shown in Definition 1 is a potential game if and only if the potential equation (5) has a solution ξ. Moreover, as (5) holds, the potential function p can be calculated by*

$$(V^p)^{\mathrm{T}} = (V_1^c)^{\mathrm{T}} - (\mathbf{1}_{k_1} \otimes I_{k^{[2,n]}})\xi_1. \tag{6}$$

3. Bi-matrix games

In this section, we consider the 2-player finite game $\mathcal{G} = (C_1, \ C_2)$, where $C_i \in \mathbb{R}^{k_1\times k_2}$ for $i = 1, 2$. In this special case, the coefficient matrices of the potential equation (5) become

$$\Psi = [-\mathbf{1}_{k_1} \otimes I_{k_2}, \ I_{k_1} \otimes \mathbf{1}_{k_2}], \qquad b = (V_2^c - V_1^c)^{\mathrm{T}}. \tag{7}$$